the general science journal

A Case-by-case Derivation of the Lorentz Transformation Equations

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May 11, 2026

Abstract

This derivation considers a train moving to the right with two unidirectional flashes of light: one originating on the train and one originating on the stationary platform. Both flashes move perpendicularly to the path of the train. Using this setup, the derivation shows that the Lorentz transformation equations are valid for all finite values of time and position. The paper also represents each of the transformation equations as a plane in a 3-dimentional space. This makes it easier to see how the times in both reference frames can be initialized to zero, even though the Lorenz transformation equations allow only one event to have a time of zero in both inertial reference frames. The 3-dimensional representations of the transformation equations can also be used to construct 2-dimensional Minkowski diagrams.

Keywords: Special Relativity, Lorentz Transformation Equations, Minkowski Diagrams

1. Introduction

The Lorentz transformation equations used in Einstein's special relativity theory have been around since at least 1905. So, we do not need to derive them again. However, in using them for a paper I was trying to write, I found that I did not understand them as well as I might like. In particular, I was worried about the initialization assumptions implied for distant sites and was not sure that the Lorentz transformation equations could be used to investigate them. So, I tried a different approach to the derivation, hoping that that would provide some insight. The result follows in sections 2 to 9.

Sections 10, 11, and 12 are appendices. In section 10, I give some background on a derivation that heavily influenced this one. In section 11, I describe another derivation, where one would naturally suppose that the domains of the transformation equations have no finite limits. Section 12 shows how a Minkowski diagram can be constructed from the 3-dimensional representations of the inverse Lorentz transformation equations.

2. The Two Simplest Cases

Consider a train, A, moving to the right in figure 0, and a stationary inertial reference frame, B, also shown. While a train is easy to imagine, we assume that it operates as if it were an inertial reference frame — i.e. no acceleration, no deceleration, no turns, no hills, no gravity, etc. Figure 0 shows the front of the train about to cross a line that is perpendicular to its path. This represents event 0. An event is defined here as an occurrence that has a single set of space coordinates and a single time coordinate in a single inertial reference frame. There would be a corresponding set of space and time coordinates for any other inertial reference frame for this same event. Since A is crossing perpendicularly to this line, we may call its crossing simultaneous to anyone at the starting line (or on a plane perpendicular to the direction of motion that contains the starting line). At the time of the crossing, all clocks on reference frames A and B are set to zero. Also at time 0, a unidirectional flash of light is emitted from the near side of the front of the train. The path of the flash is perpendicular to the path of the train and parallel to the ground (and to the plane of the tracks).

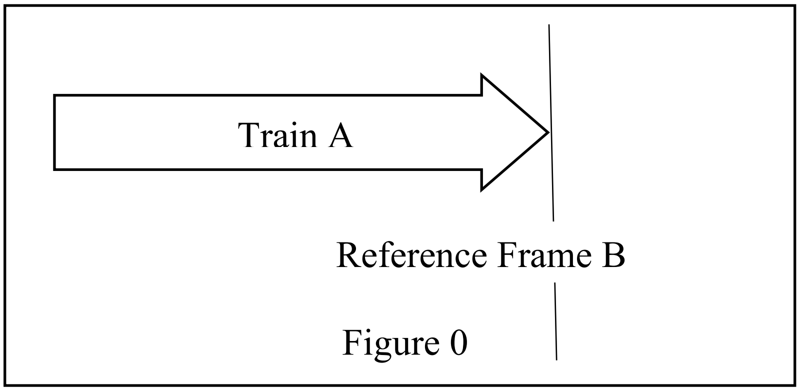

Figure 0

We assume that the origin for the reference frame of the train is at the front of the train and that the origin for frame B is the starting line. I will sometimes refer to frame B as the stationary reference frame. But this is just for reference purposes. In relativistic terms, frame B is stationary only for those on it, just as the train is stationary only for those on it.

Event 1 occurs when the flash of light reaches the other side of the train. The coordinates of event 1 are x_1', t_1' for the train and x_1, t_1 for reference frame B. The subscript 1 indicates a value for event 1. Other events will have similar subscripts. Since this is a thought experiment, we assume the width of the train could be made arbitrarily large, thereby allowing x or x' to be as large as we want. Figure 1 shows the path of that flash in reference frame B. It shows an aerial, time-lapse view of the flash (giving t vs. x) looking down on the tracks. On the right-hand side, it also shows an aerial time-lapse view of the flash (giving t' for x' = 0) looking down on the train (and moving with the train). We also assume y = y' and z = z'.

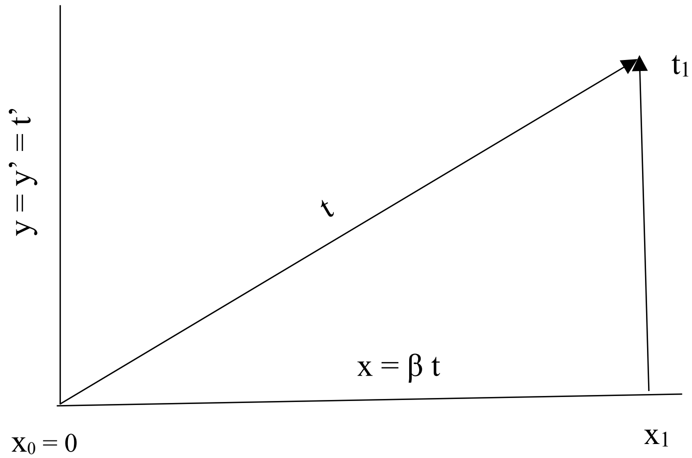

Figure 1

We also assume that both distance and time are measured in meters. Meters for time is the distance that light would travel in the interval between two events. Thus, for figure 1, y = y' = t'. β is the ratio of the speed of the train to the speed of light. β is the same for both reference frames and, by convention, is always positive. Since the origin for x' is at the front of the train, $x_1' = 0$.

The special theory also assumes the speed of light is the same in any inertial reference frame, regardless of the source of emission. So even though the flash of light was emitted from the train, we assume it is traveling at the speed of light in stationary reference frame B. As a consequence, we see in figure 1 that t is larger than t'.

Note that the time at position x_1 on the stationary platform will have been initialized to 0 during the initialization process and that the time it takes for the train to reach position x_1 is x_1/β. So $x_1 = \beta t_1$.

Using the Pythagorean theorem, we can see that:

$$t_1^2 - \beta^2 t_1^2 = t_1'^2$$

$$t_1 = t_1'/(1-\beta^2)^{1/2} \qquad (1)$$

$$x_1 = \beta\, t_1'/(1-\beta^2)^{1/2} \qquad (2)$$

Equations 1 and 2 give us part of what is needed for the Lorentz transformation equations. However, they are valid only where x' equals zero (at the front of the train). So, we need to add terms that take into account what happens when x' is not zero.

Now imagine that another unidirectional flash of light is emitted from the origin of the stationary reference frame and passes perpendicularly to the direction of motion of the train and parallel to the ground. This flash is also released at time 0 (in both inertial reference frames) from the near side of the train, but from a source on reference frame B. Event 2 occurs (still at the origin of frame B) when the flash reaches the other side of the train (or where the other side of the train once was). To avoid having the flash crash into the train, let us say the path of the flash is above the train. How that path of light would look in the reference frame of the train is shown in figure 2. It shows an aerial time-lapse view looking down on the train. The view shows where and when the flash would be seen on the train.

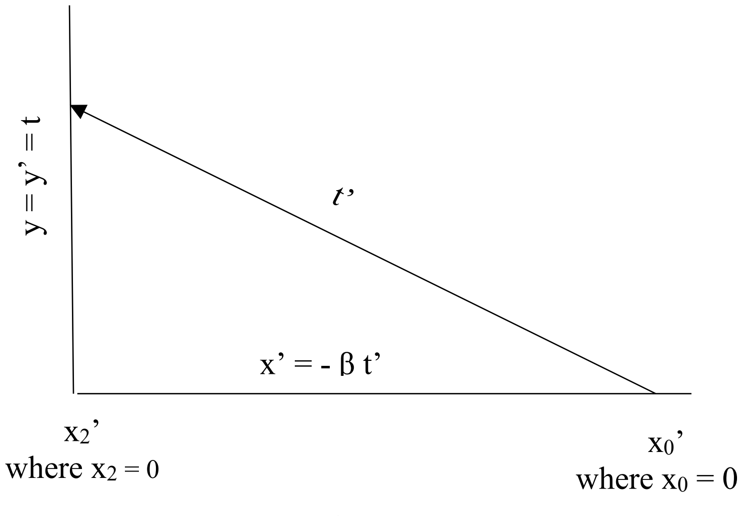

Figure 2

The coordinates for event 2 on the train are x_2', t_2'. The time it takes for a position at x_2'on the train to reach the origin of the stationary platform is $-x_2'/\beta$. Since the time at position x_2' on the reference frame of the train will have been initialized to 0 during the initialization process, $t_2' = - x_2'/ \beta$.

We are still at the origin of frame B for event 2, so x_2 must equal 0. So, what is the correction term, C, needed for equation (2) so that

$$x_2 = \beta\, t_2'/(1-\beta^2)^{1/2} + C = 0$$

It is easily seen that C must be

$$C = - \beta\, t_2'/(1-\beta^2)^{1/2} = x_2'/(1-\beta^2)^{1/2}$$

So, with this correction term equation (2) becomes

$$x_2 = x_2'/(1-\beta^2)^{1/2} + \beta\, t_2'/(1-\beta^2)^{1/2} \qquad (3)$$

Note that this equation still works at t_1', x_1', where x_1' is 0.

$$x_1 = x_1'/(1-\beta^2)^{1/2} + \beta\, t_1'/(1-\beta^2)^{1/2}$$

We also need to make a correction, A, to equation 1 so that it still holds for event 2. So, for event 2 we would have

$$t_2 = t_2'/(1-\beta^2)^{1/2} + A$$

To find A we see from figure 2 that

$$t_2'^2 - \beta^2 t_2'^2 = t_2^2$$

$$t_2^2 = (1-\beta^2)^2\, t_2'^2$$

$$t_2 = (1-\beta^2)^{1/2}\, t_2'$$

So

$$t_2 = (1-\beta^2)^{1/2}\, t_2' = t_2'/(1-\beta^2)^{1/2} + A \qquad (4)$$

This gives

$$A = t_2' \left((1-\beta^2)^{1/2} - 1/(1-\beta^2)^{1/2}\right)$$

$$A = t_2' \left((1- \beta^2 - 1)/(1-\beta^2)^{1/2}\right)$$

$$A = t_2' \left(-\beta^2)/(1-\beta^2)^{1/2}\right)$$

But - β t_2’ = x_2’.

So, $$A = \beta x_2'/(1-\beta^2)^{1/2}$$

Substituting A into equation 4 we have

$$t_2 = \beta x_2'/(1-\beta^2)^{1/2} + t_2'/(1-\beta^2)^{1/2} \quad (5)$$

If we remove the subscripts, this is the usual form of the Lorentz transformation equation for time. It also works for event 1 (with a suitable change of subscripts), where x_1’ = 0. We might also want to show that this same equation along with that of equation 3 are true for events unlike events 1 or 2 — that is, when we are neither at the front of the train nor at the origin of frame B. However, it is not clear that this is possible. Also, we must discuss the initialization assumptions, since we have not so far shown that we can initialize the times of both t and t’ to zero.

3.0 The Initialization Assumptions

It is, in theory, possible to synchronize the clocks on a single inertial reference frame by sending a starting time from a standard clock to distant points. The clocks at the distant locations would set their times to the starting time plus the transmission time. Once this is done, clocks on such an inertial reference frame could be reset to zero (or some other value) at a prearranged time. Or they could be changed at leisure, so long as the time reflected the new universal time for that inertial reference frame. Or we, as outside observers, could simply assume such changes had been made for our theoretical purposes.

The Lorentz transformation equations assume that we initialize the times and the distances for all locations that will be needed for any upcoming calculations. The distances and times need to be set relative to event 0, where x’ = x = t’ = t = 0. This sounds simple, since both distances and times are additive (or subtractive) within any given inertial reference frame — just as they are in pre-relativity physics.

However, when two inertial reference frames are involved and we are using the Lorentz transformation equations, the situation is more complex. The times in both inertial reference frames could still be initialized to zero. However, from the point of view of a either reference frame, only a single event can be time zero in both reference frames. This is shown in equation 5a below (which is the same as equation 5, but with the subscripts removed). For future use, I have also included equation 3a below (which is the same as equation 3 with the subscripts removed).

$$x = x'/(1-\beta^2)^{1/2} + \beta t'/(1-\beta^2)^{1/2} \quad (3a)$$

$$t = \beta x'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2} \quad (5a)$$

Let us assume, for this section, that equations 3a and 5a are valid for all finite values of x' and t'. (We have not yet proven this, but will do so in coming sections.) From equation 5a we see that no event, other than event 0, can have zero for both t and t'. Equation 5a does permit us to set all values of t to zero. But except for event 0, this must occur when $t' \neq 0$ and when

$$\beta x'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2} = 0. \qquad (6)$$

To clarify matters, it is helpful to realize that equation 5a can be represented by a plane in a 3-dimensional space with, for example, x' on the X axis, t' on the Y axis, and t on the Z axis. The plane of 5a will pass though the origin, where x', t', and t are all zero. There will be a line where the plane of equation 5a intercepts the horizontal plane where $t = 0$. It satisfies the constraint of equation 6 that: $t' = -\beta x'$. This is shown in figure 3 below.

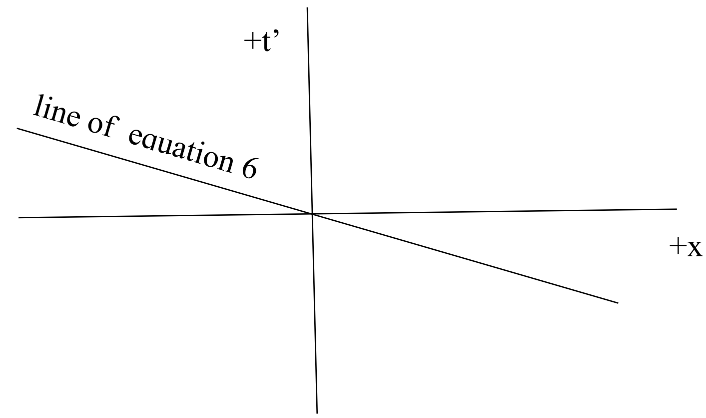

Figure 3

As a precursor to event 1 we are interested in the event (which I will call event 3) when the time at x_1 is initialized to zero. If we are consistent with the Lorentz transformation equations and t is zero, event 3 would appear somewhere on the line of equation 6. For positive values of x', equation 6 requires negative values of t'. So event 3 would appear on the equation of line 6 in the fourth quadrant of figure 3. So for event 3, t' is negative.

Similarly, we could have an abstract space that contains the plane of equation 3a, with x on the Z axis, rather than t. We can imagine a horizontal plane in this space where $x = x_1$. This plane would intercept the plane of 3a in a line. Event 3 would also be represented as a point on that line. If we set x to x_1 in equation 3a and t to zero in equation 5a, we can solve for the values x' and t' that will be present at this initialization event.

We can imagine a 2-dimentional world line (in the abstract space where the plane of equation 3a intercepts the horizontal plane where $x = x_1$) that follows the position of x_1 after its time is initialized to zero. As t' increases, equation 3a requires that x' decrease, in order to keep a constant value of x. Eventually, the world line for x_1 will arrive at an event where x' equals zero. This is event 1, when the front of the train reaches x_1. Since the time at x_1 will have been initialized to zero, and since the front the train has passed a distance of x_1 in time t_1 (according to the stationary reference frame), it follows that $x_1 = \beta t_1$ — as we assumed in figure 1. So there is no problem in initializing t_1 to zero.

Going back to the 3-dimentional space that includes the plane of equation 5a, the initialization of time to zero on the inertial reference frame of the train can be represented as events on a plane that includes the x' axis and is perpendicular to the horizontal plane. There all t' is zero. This plane will intercept the plane of equation 5a directly below the x' axis where x' is negative, and directly above the x' axis where x' is positive. Any point on this intercept will satisfy equation 5a, since it is on the plane of equation 5a.

Any point on that intercept line where x' has a negative value could show up as an event like event 2. In particular, we want to initialize the time at x_2' to zero. Let us call that initialization event, event 4. Also let us imagine that it is at the end of the train. Doing so allows the end of the train to show up with a time that satisfies $x_2' = -\beta t_2'$, as was assumed in figure 2. Doing so does not contradict anything we have said so far. Notice, however, that event 4 occurs on the boundary between the second and third quadrant of figure 3. At this boundary, if you are on the plane of equation 5a, t is negative.

There could be other points in the spaces where the planes of equations 3a and 5a occur, that are not on the planes of those equations. If we assume special relativity applies here, those off-the-plane points, would presumably increment (or decrement) in a plane that is parallel to the plane of equation 3a or 5a. For example, they could represent events associated with locations which have not been properly initialized. I will discuss these in the next section

So let us summarize our assumptions. First, we assume that the laws are physics are the same in all inertial reference frames. Second, we assume that the speed of light is the same in all inertial reference frames, regardless of the source of emission of that light. Third, we assume that any two considered reference frames share a common origin, such as event 0 (where $x' = y' = z' = t' = x = y = z = t = 0$). Fourth, we assume for all events (not just event 0) that $y = y'$ and $z = z'$. Fifth, we assume that time is universally synchronized within any considered inertial reference frame. And sixth, β cannot equal 1, since that would involve dividing by zero in the Lorenz transformation equations. The first two assumptions are from Einstein. The third and fourth are usually stated. And the fifth and sixth are implicitly assumed, if not stated.

3.1 What happens if the inertial reference frames are not properly initialized?

Remember, we assume that each individual reference frame has a universal time. By that I mean that each has synchronized clocks, as described in section 3. That means there once was a time when it was time zero throughout the train. So, since the origin of the train (at its front) once had a time of zero, the train is properly initialized (and had an event 0) simply by having a universal time.

Suppose at that event 0 we also initialized the coordinates for the stationary reference frame, but we kept the old times and positions as a back-up. If we use the new coordinates for the stationary reference frame, equations 3a and 5a are valid (at least for the two cases described in section 2).

But we can modify equation 3a and 5a, to get the old coordinates. This is as follows.

$$x_{new} = x'/(1-\beta^2)^{1/2} + \beta\, t'/(1-\beta^2)^{1/2} \qquad (3a)$$

$$t_{new} = \beta\, x'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2} \qquad (5a)$$

$$A = x_{new} - x_{old}$$

$$B = t_{new} - t_{old}$$

$$x_{new} = x_{old} + A$$

$$t_{new} = t_{old} + B$$

$$x_{old} = x'/(1-\beta^2)^{1/2} + \beta\, t'/(1-\beta^2)^{1/2} - A \qquad (3b)$$

$$t_{old} = \beta\, x'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2} - B \qquad (5b)$$

Note that equation 5b can be represented by a plane in a 3-dimensional space with, for example, x' on the X axis, t' on the Y axis, and t on the Z axis. Equation 5a could appear in the same space and would be a plane parallel to the plane of equation 5b. The plane of 5b, would not, of course, pass through the origin of the 3-dimentional space, while the plane of 5a would.

Similarly, equation 3b can be represented by a plane in a 3-dimensional space with, for example, x' on the X axis, t' on the Y axis, and x on the Z axis. Equation 3a could appear in the same space and would be a plane parallel to the plane of equation 3b. The plane of 3b, would not pass through the origin of the 3-dimentional space, while the plane of 3a would.

After this discussion of initialization assumptions, we still need to establish whether equation 3a and 5a are valid for events neither at the front of the train nor at the origin of the stationary reference frame. So let us return to that discussion.

4. An initialization event where t' is set to zero for some x' < 0

We are calling the event where the clock at x_2' is initialized to zero, event 4. Even though we have shown in section 3 that event 4 is consistent with the Lorentz transformation equations, we have not shown that its positions on the planes of equations 3a and 5a follow directly from special relativity. So that is considered here.

Suppose that at the time of initialization event 4 was like event 0. Then we could use the same flashes of light and discussion we used for deriving equations 3 and 5. Then the arrival of x_4’ at the origin of the stationary platform would be like event 1. Let us call that arrival event 5. So, using equations 1 and 2, for event 5 we would have

$$t_5 = t_5'/(1-\beta^2)^{1/2}$$
$$x_5 = \beta\, t_5'/(1-\beta^2)^{1/2}$$

Now let us suppose that event 5 occurs at the end of the train. Then as far the train is concerned, it takes the same amount of time for the front of the train to go to event 1 as for the end of the train to go to event 5. Then $t_5' = t_2' = t_1'$. Also, in terms of value, $x_5' = x_4' = x_2'$.

But using our previous assumptions, event 5 would occur at the same time and place as event 2. And for event 2 we know that

$$t_2 = \beta\, x_2'/(1-\beta^2)^{1/2} + t_2'/(1-\beta^2)^{1/2}$$
$$x_2 = x_2'/(1-\beta^2)^{1/2} + \beta\, t_2'/(1-\beta^2)^{1/2}$$

This leaves us with two correction terms for event 4 to correct for the difference between the event 2 assumptions and the event 5 assumptions:

$$t_4 = t_0 + A \qquad A = t_2 - t_5$$
$$x_4 = x_0 + B \qquad B = x_2 - x_5$$

$$A = \beta\, x_2'/(1-\beta^2)^{1/2} = \beta\, x_4'/(1-\beta^2)^{1/2}$$
$$B = x_2'/(1-\beta^2)^{1/2} = x_4'/(1-\beta^2)^{1/2}$$

$$t_4 = t_0 + A = t_0 + \beta\, x_4'/(1-\beta^2)^{1/2}$$
$$x_4 = x_0 + B = x_0 + x_4'/(1-\beta^2)^{1/2}$$

And since both t_0 and x_0 equal zero, we have

$$t_4 = \beta\, x_4'/(1-\beta^2)^{1/2}$$
$$x_4 = x_4'/(1-\beta^2)^{1/2}$$

And since t_4’ has been initialized to zero, we may add t_4’ terms to these equations giving:

$$t_4 = \beta\, x_4'/(1-\beta^2)^{1/2} + t_4'/(1-\beta^2)^{1/2}$$
$$x_4 = x_4'/(1-\beta^2)^{1/2} + \beta\, t_4'/(1-\beta^2)^{1/2}$$

So the Lorentz transformation equations (3a and 5a) can be used for any negative value of x’ where t’ = 0. And we also know, by our initialization assumptions, that

3a and 5a are valid for event 0. Consequently, 3a and 5a are valid whenever $t' = 0$ and $x' \leq 0$.

5. An initialization event where t' is set to zero for some x' > 0

Let us define event 6 to be an initialization event where we set t' to zero for some $x' > 0$. Our basic strategy here is to imagine that at time zero on the train, x_6 is across from the front of a second train, which has the same length as the first. This train precedes our previously-discussed train with no intervening separation.

When the front of the trailing train (the one that left x_0 at t_0') reaches event 1, the end of the leading train will also be at event 1. So event x_6 occurs where (but not when) x_1 occurs on the stationary reference plane

Now imagine that we treat event 6 as if it were event 0. Let event 7 be the event that occurs when the end of the leading train reaches x_1 in this false narrative. Then under the false narrative event 7 would be like event 2. So

$$t_7 = \beta\, x_7'/(1-\beta^2)^{1/2} + t_7'/(1-\beta^2)^{1/2}$$
$$x_7 = x_7'/(1-\beta^2)^{1/2} + \beta\, t_7'/(1-\beta^2)^{1/2}$$

But for the true narrative we know from equations 1 and 2 that

$$t_1 = t_1'/(1-\beta^2)^{1/2}$$
$$x_1 = \beta\, t_1'/(1-\beta^2)^{1/2}$$

So we can get t_6 and x_6 from

$$t_6 = t_0 + A \qquad A = t_1 - t_7$$
$$x_6 = x_0 + B \qquad B = x_1 - x_7$$

And since from the point of view of the trains, it takes the same amount of time for the front of the trailing train to go from event 0 to event 1 as for the end of the leading train to traverse this identical path at the same time, $t_1' = t_7'$. And for the false narrative $x_7' = -x_6'$, since x_7' is at the back of the first train. This results in

$$A = -\beta\, x_7'/(1-\beta^2)^{1/2} = \beta\, x_6'/(1-\beta^2)^{1/2}$$
$$B = -x_7'/(1-\beta^2)^{1/2} = x_6'/(1-\beta^2)^{1/2}$$

$$t_6 = t_0 + A = t_0 + \beta\, x_6'/(1-\beta^2)^{1/2}$$
$$x_6 = x_0 + B = x_0 + x_6'/(1-\beta^2)^{1/2}$$

And since both t_0 and x_0 are zero

$$t_6 = \beta\, x_6'/(1-\beta^2)^{1/2}$$
$$x_6 = x_6'/(1-\beta^2)^{1/2}$$

And since $t_6' = 0$, we may add its terms.

$$t_6 = \beta\, x_6'/(1-\beta^2)^{1/2} + t_6'/(1-\beta^2)^{1/2}$$
$$x_6 = x_6'/(1-\beta^2)^{1/2} + \beta\, t_6'/(1-\beta^2)^{1/2}$$

So event 6 follows equations 3a and 5a. And since in earlier sections we have shown that equations 3a and 5a are valid for $x' \leq 0$ and $t' = 0$, we can now claim equations 3a and 5a are valid for $-\infty < x' < \infty$, assuming $t' = 0$. Events where t' does not equal zero are treated in the next section.

6. Events on the train where time is not zero

To give a geometric interpretation of the events where t' is not zero, imagine a plane that is perpendicular to the horizontal plane, parallel to the t' axis, and passes through a particular value to x'. The intercepts of this plane with the 5a plane in one space and with the 3a plane in a second space give the world line (for t) and something similar (for x) for that value of x' when t' changes. For 5a, increasing (or decreasing) values of t' affect t, according to $t = t'/(1-\beta^2)^{1/2}$. For 3a, increasing (or decreasing) values of t' affect x, according to $x = \beta\, t'/(1-\beta^2)^{1/2}$.

To show that 5a and 3a are consistent with special relativity for positive values of t', recall the differences in t and x between event 0 and event 1. For event 1, $t_1 = t_1'/(1-\beta^2)^{1/2}$ and $x_1 = \beta\, t_1'/(1-\beta^2)^{1/2}$. Now consider what is the difference between any other two events, let's say events 8 and 9, where $t_9' - t_8' = t_1' - t_0'$, $t_8' = t_0' = 0$, and $x_9' = x_8'$ (just as $x_1' = x_0'$). We can imagine x_9' as having passed a stretch on the stationary platform in the same amount of time (on a train that precedes the event-0-and-event-1 train) as x_1' needs to get to event 1 from event 0. Anyone on either train would agree that an observer at x_9' has passed the same distance and time between events 8 and 9 as the observer at x_1' has passed between event 0 and 1.

Further we could use the same flash of light used in figure 1, and would get the same results, namely:

$$t_1 = t_1'/(1-\beta^2)^{1/2}$$
$$x_1 = \beta\, t_1'/(1-\beta^2)^{1/2}$$

But for event 9, these amounts need to be increased by the starting amounts at event 8.

$$t_9 = t_8 + t_1'/(1-\beta^2)^{1/2}$$
$$x_9 = x_8 + \beta\, t_1'/(1-\beta^2)^{1/2}$$

And since $t_9' = t_1'$,

$$t_9 = t_8 + t_9'/(1-\beta^2)^{1/2}$$
$$x_9 = x_8 + \beta\, t_9'/(1-\beta^2)^{1/2}$$

And since $t_8' = 0$ and $x_8' = x_9'$

$$t_8 = \beta\, x_8'/(1-\beta^2)^{1/2} = \beta\, x_9'/(1-\beta^2)^{1/2}$$
$$x_8 = x_8'/(1-\beta^2)^{1/2} = x_9'/(1-\beta^2)^{1/2}$$

Consequently,

$$t_9 = \beta\, x_9'/(1-\beta^2)^{1/2} + t_9'/(1-\beta^2)^{1/2}$$
$$x_9 = x_9'/(1-\beta^2)^{1/2} + \beta\, t_9'/(1-\beta^2)^{1/2}$$

With the subscripts removed, these two equations are the same as equations 5a and 3a. So considering what we have previously shown, we have now shown that the domain where these equations are consistent with special relativity is: $0 \leq t' < \infty$ and $-\infty < x' < \infty$.

For negative values of t' we can imagine an event, event 10, which is a precursor to some event, event 11, where $t_{11}' = 0$ and $x_{10}' = x_{11}'$. The difference between event 10 and event 11 is like that between event 0 and event 1. However, since $t_{11}' = 0$, t_{10}' must have a negative value. Again, we use event 1.

$$t_1 = t_1'/(1-\beta^2)^{1/2}$$
$$x_1 = \beta\, t_1'/(1-\beta^2)^{1/2}$$

But in this case, we must subtract the event 1 changes from event 11 to get the event 10 values. For this we begin with equations 3a and 5a, noting, however, that $t_{11}' = 0$.

$$x_{11} = \beta\, t_{11}'/(1-\beta^2)^{1/2} + x_{11}'/(1-\beta^2)^{1/2} = x_{11}'/(1-\beta^2)^{1/2})$$
$$t_{11} = t_{11}'/(1-\beta^2)^{1/2} + \beta\, x_{11}'/(1-\beta^2)^{1/2} = \beta\, x_{11}'/(1-\beta^2)^{1/2}$$

And since $x_{10} = x_{11} - x_1$, and $t_{10} = t_{11} - t_1$,

$$x_{10} = x_{11}'/(1-\beta^2)^{1/2} - \beta\, t_1'/(1-\beta^2)^{1/2}$$
$$t_{10} = \beta\, x_{11}'/(1-\beta^2)^{1/2} - t_1'/(1-\beta^2)^{1/2}$$

And since $t_1' = -t_{10}'$ and $x_{11}' = x_{10}'$,

$$x_{10} = x_{10}'/(1-\beta^2)^{1/2} + \beta\, t_{10}'/(1-\beta^2)^{1/2}$$
$$t_{10} = \beta\, x_{10}'/(1-\beta^2)^{1/2} + t_{10}'/(1-\beta^2)^{1/2}$$

So again, if we remove the subscripts, these are the same as equations 3a and 5a. And we can now say that equations 3a and 5a are consistent with special relativity when $-\infty < t' < \infty$ and $-\infty < x' < \infty$.

7. The inverse Lorentz transformation equations

The inverse Lorentz transformation equations, equations 3c and 5c below, are the opposite of equations 3a and 5a. They go from the stationary platform coordinates to the train (or rocket) coordinates, rather than the other way around.

$$x' = x/(1-\beta^2)^{1/2} - \beta\, t/(1-\beta^2)^{1/2} \qquad (3c)$$
$$t' = -\beta\, x/(1-\beta^2)^{1/2} + t/(1-\beta^2)^{1/2} \qquad (5c)$$

The second edition of Taylor/Wheeler (1992, p. 103) provides two short derivations of the inverse Lorentz transformation equations. One proof is purely algebraic. It solves for two unknowns in two equations. The other proof is based on a symmetry argument. Below I will use the first approach.

First multiply equation 3a on both sides by β and subtract that from 5a.

$$x = x'/(1-\beta^2)^{1/2} + \beta\, t'/(1-\beta^2)^{1/2} \qquad (3a)$$
$$t = \beta\, x'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2} \qquad (5a)$$
$$\beta x = \beta x'/(1-\beta^2)^{1/2} + \beta^2\, t'/(1-\beta^2)^{1/2}$$
$$t - \beta x = (1-\beta^2)\, t'/(1-\beta^2)^{1/2} = (1-\beta^2)^{1/2}\, t'$$

So $$t' = -\beta\, x/(1-\beta^2)^{1/2} + t/(1-\beta^2)^{1/2} \qquad (5c)$$

Next multiply 5a on both sides by β and subtract that from 3a.

$$x - \beta\, t = (1-\beta^2)\, x'/(1-\beta^2)^{1/2} = (1-\beta^2)^{1/2}\, x'$$

So $$x' = x/(1-\beta^2)^{1/2} - \beta\, t\, /(1-\beta^2)^{1/2} \qquad (3c)$$

Since the non-inverse Lorentz transformation equations are valid for $-\infty < x' < \infty$ and $-\infty < t' < \infty$, the inverse transformation equations are valid for $-\infty < x < \infty$ and $-\infty < t < \infty$. This is true because of the reciprocal relationship between the inverse and non-inverse transformations. If you know x' and t', you can always find x and t, and vice versa. Because of this, it is always possible to find a pair of values x_{test}', t_{test}' (on the right-hand side of equations 3a and 5a) that give the coordinates x_{test} and t_{test} (on the left-hand side of equations 3a and 5a). This can be done using two equations in two equations (or using equations 3c and 5c). Since x_{test} and t_{test} could be any pair of finite values, that is the domain of equations 3c and 5c.

8. Interpreting the equations

Even without the above, we can see that equations 3a and 5a have an obvious interpretation. The terms in x' give the appropriate values for x and t when t' is zero. Any nonzero terms in t' are adjustments for what happens after (or before) t' is zero.

$$x = x'/(1-\beta^2)^{1/2} + \beta\, t'/(1-\beta^2)^{1/2} \qquad (3a)$$
$$t = \beta\, x'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2} \qquad (5a)$$

Another way of looking at the equations takes into account how they were derived. For event 1, x' is still zero and we are at the front of the train. In that case, x is larger than an observer at the front of the train thinks he/she has traveled. In other words: $x = \beta\, t'/(1-\beta^2)^{1/2} > \beta\, t'$. Also $t = t'/(1-\beta^2)^{1/2}$. So t > t' by the same factor.

On the other hand, if we remain at the origin of the stationary platform, $x' = -\beta\, t'$ according to equations 3a. If we substitute this for x' in equation 5a, we have

$$t = -\beta^2\, t'/(1-\beta^2)^{1/2} + t'/(1-\beta^2)^{1/2}$$

So

$$t = (1-\beta^2)\, t'/(1-\beta^2)^{1/2} = t'(1-\beta^2)^{1/2}$$

So when the end of the train reaches the origin of the stationary platform, its time will be longer than that of the stationary platform. Also its distance traveled, $\beta\, t'$, will be longer than $\beta\, t$ by the same factor $(1/(1-\beta^2)^{1/2})$. This is just the opposite of that for the front of the train. But it makes intuitive sense. In relativistic terms, the origin of the stationary platform has moved toward the end of the train in the same way that the front of the train has moved toward event 1.

Also in this case, two clocks on the train (one at the front and one at the back) were involved and only one on the stationary platform. Whereas in the changes from event 0 to event 1, two clocks on the stationary platform were involved (one at the origin and one at x_1), but only one on the train. As discussed in section 3, in both initialization events (event 3 and event 4), when (and where) the second clock was initialized to zero, the event-sharing (but uncounted) clock on the other inertial reference frame was negative.

9. Summary and discussion

The main goal of this derivation was to show that the transformation equations are consistent with special relativity for all finite values of time and position. This is true both for the first set of transformation equations (those going from the train coordinates to the stationary-platform coordinates) and the inverse transformation equations (those going the other way).

A second feature of the paper is the representations of the transformation equations as planes in 3-dimentional spaces. From this we see how it is possible to initially set all times in both inertial reference frames to zero, even though only one event can have a time of zero in both reference frames. Also, we see what happens when the time and position are not properly initialized in both reference frames. The usual Lorentz transformation equations no longer work, but some more general ones do.

Also of interest is an interpretation of the terms in the transformation equations (as discussed in section 8). The terms in x' give the values of x and t when t' is zero. The terms in t' show how x and t are affected when t' increases (or decreases). While this is obvious if you look at the Lorentz transformation equations themselves, here it is an integral part of the derivation.

Also in section 8, we can see in relativistic terms, that the origin of the stationary platform has moved toward the end of the train in the same way that the front of the train has moved toward to a place on the stationary platform (where event 1 occurs). So, when the end of the train reaches the origin of the stationary platform, its elapsed time will be longer than that of the stationary platform.

So it is not always the stationary platform that has the larger elapsed time between two events. It is the inertial reference frame where two clocks are involved. The second clock is the one that was not present at the origin when its time was initialized to zero. As discussed in section 3, in both initialization events (event 3 and event 4), when (and where) the second clock was initialized to zero, the event-sharing (but uncounted) clock on the other inertial reference frame was negative.

Also, we have assumed no reflection (or passing through glass) of the flashes of light. Consequently, we have made no assumption about what happens during such interventions.

And finally, in section 12, I show how a Minkowski diagram can be constructed by projecting the two 3-dimensional representation of the inverse Lorenz transformation equations onto the plane containing the x and t axes. This shows that the x' and t' axes have the same scale, as do the x and t axes. However, the scaling factors for the x' and t' axes do not equal those for the x and t axes.

10. The Tayler-Wheeler derivations

My first exposer to the equations was from Taylor's and Wheeler's textbook *Spacetime Physics* (1966, pp. 39-44). My equations based on t' alone (equations 1 and 2 in section 2 above) were developed by Taylor and Wheeler using the invariance of the spacetime interval ($t'^2 - x'^2 = t^2 - x^2$). However, they justify the invariance of the spacetime interval only for intervals where $t^2 > x^2$.

For the next step, Taylor and Wheeler assumed that the equations are a simple transformation of coordinates and take the following form:

$$t = (1 - \beta^2)^{-1/2}\, t' + A\, x'$$
$$x = \beta\, (1 - \beta^2)^{-1/2}\, t' + B\, x'$$

The constants A and B are found by substituting the right-hand sides of the above two equations for t and x into the left-hand side of the following equation for the constancy of the spacetime interval:

$$t^2 - x^2 = t'^2 - x'^2$$

This approach is straight-forward, but are we justified in assuming the final form of the equations? Also, since the constancy of the spacetime interval was not established for cases where $x^2 > t^2$, do we need an independent proof of it if we are to use the Lorentz transformation equations for such cases?

If the linearity assumption is solid, the coefficients of the independent variables are the same for all events, and nothing else is required, the answer is “no.” In other words, we can use special cases to solve for the coefficients of the independent variables (x’ and t’), if we are sure that the coefficients do not vary from event to event.

A second edition of Taylor and Wheeler’s textbook, *Spacetime Physics an introduction to special relativity (1992, p. 95-103)*, uses the same approach as before. However, they give a good explanation why the form should be linear. Otherwise, events that are equally spaced (in either position or time) in one inertial reference frame would not be equally spaced in a second. They also point out that the time and position of event 0 (where x = x’ = t = t’ = 0) is arbitrary. Consequently, we would not expect the coefficients to vary either.

However, there is one further wrinkle. As I discussed in section 3.1, a pair of intercept constants are needed when the two inertial reference frames do not share a common origin for time and distance. Although we always end up using a common origin, when you argue that the time and position of even 0 is arbitrary, that suggests that intercept constants might be needed. In fact, there are points in the derivation in sections 4 and 5, where something like the constants A and B of section 3.1 are needed. These turn out to equal the ordinary coefficients for x’, so no intercept adjustments are needed.

11. The A. P. French derivation

The A. P. French derivation (1968, pp. 74-81) is based on a series of Minkowski diagrams, along with a linearity assumption and a symmetry argument. French develops these diagrams using the assumptions of special relativity. The Lorentz transformation equations are then partly based the diagrams.

Since the diagrams have values for x, x’, t, and t’ for all finite values of these variables in a two-dimensional diagram, it seems reasonable to suppose that the Lorentz transformations would be valid for all the possible combinations of the variables. However, French does not prove this (or claim such a proof). So, we may have some lingering doubts about the applicability of the Lorenz transformations when either x’ or t’ has something unexpected, such as a negative value.

In any case, if x, x’, t and t’ are all measured in meters (or some other consistent measure), the Minkowski diagram will have the same scale for x and t. However, they will not have that same scale for x’ and t’. This feature of the Minkowski diagrams is explored further in the next section.

12. Constructing a Minkowski Diagram Using Equations 3c and 5c

Equations 3c and 5c, shown below, are the inverse Lorentz transformation equations discussed in section 7. They go from the stationary platform coordinates to the train (or rocket) coordinates, rather than the other way around.

$$x' = x/(1-\beta^2)^{1/2} - \beta\, t/(1-\beta^2)^{1/2} \qquad (3c)$$
$$t' = -\,\beta\, x/(1-\beta^2)^{1/2} + t/(1-\beta^2)^{1/2} \qquad (5c)$$

Let us represent equation 5c as a plane in a 3-dimensional space with x on the X axis, t on the Y axis, and t’ on the Z axis. Let us represent equation 3c as a plane in that same 3-dimensional space, with x’ also measured on the Z axis. If x, t, x’, and t’ are all measured in meters, all distances in that space have the same scale.

A Minkowski diagram can be constructed simply by projecting points from these two planes onto the horizontal plane containing the origin and the x and t axes. The points are projected along a line that is perpendicular to any horizontal plane.

The simplest of the projections are not really projections, in the full sense of the word. In the horizontal plane at the origin, the lines of t’ and x’ for figure 4 are the intercepts of the planes of equations 3c and 5c with that horizontal plane. The equations for these lines are shown below.

$$x = \beta\, t \qquad (3c.0)$$
$$\beta\, x = t \qquad (5c.0)$$

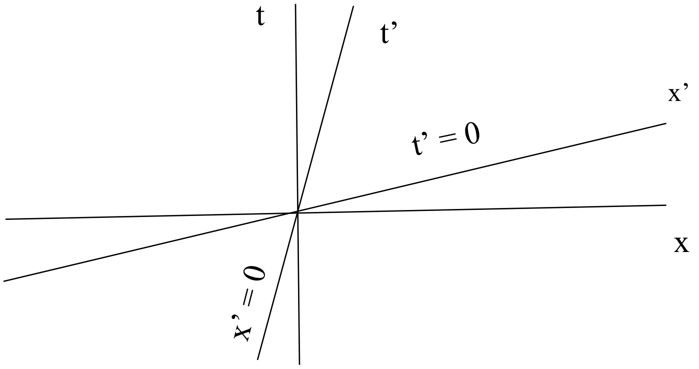

Figure 4

The line marked x’ = 0 is the intercept of the 3c plane with the horizontal plane. On this line $x = \beta t$. The line in figure 4 marked t’= 0, is the intercept of the 5c plane with the horizontal plane. On this line $t = \beta x$. The line where x’ = 0 represents the t’ axis, while the line where t’ = 0 is the x’ axis. As will be discussed, x’ and t’ do not have the same scale as x and t.

Figure 5 adds a line projected from the plane where x’ equals some, fixed nonzero value, k_{12}. Figure 5 also adds a line projected from the plane where t’ equals some fixed, nonzero value, k_{13}. The first is projected from the line where the plane of equation 3c intercepts the horizontal plane where x’ equals k_{12}. The second is projected from the line where the plane of equation 5c intercepts the horizontal plane where t’ equals k_{13}. They are given by the following equations in the shared 3-dimentional space.

$$k_{12} = x/(1-\beta^2)^{1/2} - \beta\, t/(1-\beta^2)^{1/2} \qquad \text{(3c.k12)}$$
$$k_{13} = -\beta\, x/(1-\beta^2)^{1/2} + t/(1-\beta^2)^{1/2} \qquad \text{(5c.k13)}$$

The derivative, dt/dx, of equation 3c.k12 is: $dt/dx = 1/\beta$. This derivative is the same as that of equation 3c.0. So its projection onto the horizontal plane at the origin, on which figure 5 is based, will be parallel to that of the t’ axis in figure 5.

The derivative, dt/dx, of equation 5c.k13 is: $dt/dx = \beta$. This derivative is the same as that of equation 5c.0. So its projection onto the horizontal plane at the origin, on which figure 5 is based, will be parallel to that of the x’ axis in figure 5.

All events on the x’ axis of figure 5 have the same value for t’, namely 0. Similarly, all events on the line marked t’ = k_{13} have the same value for t’, namely k_{13}. Also, all events on t' axis have the same value for x', namely 0, and all events on the line marked x' = k_{12} have the same value for x', namely k_{12}. Both k_{12} and k_{13} can have any finite value. And, of course, there could be any number of similar lines.

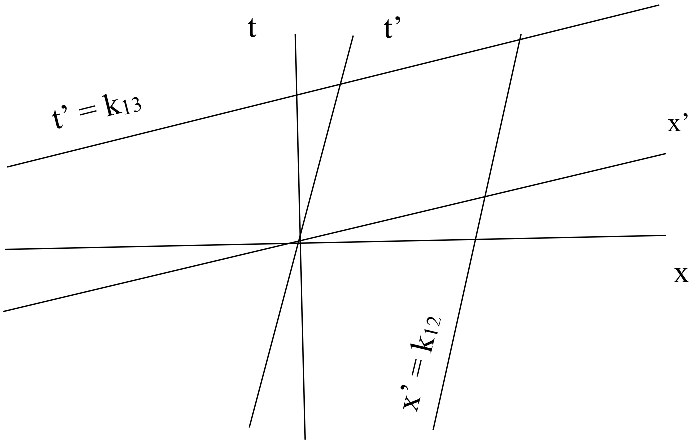

Figure 5

The world line of the point where x' = 0, is the t' axis. Similarly, the world line of the point where x' = k12 is parallel to the t' axis and passes through the point x' = k12 on the x' axis.

There are no distortions of t and x in creating figure 5. They are on the horizontal plane that contains the origin of our 3-dimentional space. While t' and x' are measured on the Z dimension of the 3-dimentional space from which we are projecting, the distance from the origin to some point (x', t') in figure 5 looks like it should equal $(x^2 + t^2)^{1/2}$. So it must have a different scale.

To find that scale, suppose the lengths on the page we see on the x or t axes were obtained by multiplying the represented real-world distances by their scaling factors, SFx and SFt. These scaling factors, we assume, are equal and given as the ratio of the length on the page to the represented distance in the world. What we want to find are the corresponding scaling factors for x' and t', SFx' and SFt'.

We want the represented distances of t' in diagrams 4 and 5 to be given by equation 5c. Since the lengths from the origin on the page are the same, we can say for SFt' that

$$SFt\,(x^2 + t^2)^{1/2} = SFt'(-\beta x/(1-\beta^2)^{1/2} + t/(1-\beta^2)^{1/2})$$

Or

$$SFt' = \frac{SFt\,(x^2 + t^2)^{1/2}}{(-\beta x/(1-\beta^2)^{1/2} + t/(1-\beta^2)^{1/2})} \qquad (7)$$

Similarly, we want the represented distances of x' in diagrams 4 and 5 to be given by equation 3c. But the apparent length on the page equals SFx $(x^2 + t^2)^{1/2}$. So the scaling factor SFx' is given by

$$SFx' = \frac{SFx\ (x^2 + t^2)^{1/2}}{(x/(1-\beta^2)^{1/2} - \beta\ t/(1-\beta^2)^{1/2})} \qquad (8)$$

However, $x = \beta t$ when we are on the t' axis. So for these cases, equation 7 simplifies to

$$SFt' = \frac{(\beta^2 t^2 + t^2)^{1/2}\ SFt}{t\ (1- \beta^2)/(1-\beta^2)^{1/2}} = \frac{(1+ \beta^2)^{1/2}}{(1- \beta^2)^{1/2}}\ SFt \qquad (9)$$

Also, $t = \beta x$ when we are on the x' axis. So for these cases, equation 8 simplifies to

$$SFx' = \frac{(x^2 + \beta^2 x^2)^{1/2} SFx'}{x\ (1 - \beta^2)/(1-\beta^2)^{1/2}} = \frac{(1 + \beta^2)^{1/2}}{(1 - \beta^2)^{1/2}}\ SFx \qquad (10)$$

And since we assumed SFx = SFt, we can say for these cases and for $0 \leq \beta < 1$ that

$$SFx = SFt \leq SFt' = SFx' \qquad (11)$$

And that means that the represented distances on the x' and t' axes are smaller than they look, as compared to distances on the x or t axis. An exception occurs when β equals zero and SFt = SFt'. Also note that β must be less than 1. If β were allowed to equal 1, SFt' and SFx' would be infinite in equations 9 and 10 respectively. This is not an additional constraint, however, since none of the Lorentz transformation equations make sense if β is 1 or larger.

Also note that we have shown only how to find SFt' and SFx' for the t' and x' axes. That is, nevertheless, sufficient. The opposite sides of the parallelogram shown in figure 5 have the same lengths on the page. And the represented time, t', is the same along any line of constant time. Consequently, SFt' also applies to any point on line parallel to the t' axis. Similarly, SFx' also applies to any point on line parallel to the x' axis.

References

1. French, A. P.; *Special Relativity*; W.W. Norton & Co. (New York, 1968)
2. Taylor, E. F. and Wheeler, J. A.; *Spacetime Physics*; W.H. Freeman and Co. (San Francisco, 1966)
3. Taylor, E. F. and Wheeler, J. A.; *Spacetime Physics, second edition*; W.H. Freeman and Co. (San Francisco, 1992)