the general science journal

New geometrical derivations of Machin type equation

Simona A. Popescu and Aaron Peled*
Engineering Faculty, Holon Institute of Technology
52 Golomb Str. Holon 5867921, Israel

*Corresponding Author
Email: peledd@gmail.com

Abstract

Two Machin type equations were derived by summing arctangent functions giving π. New triangles constructions on a Cartesian squares lattice were used to obtain by induction these equations, as combinations of single variable functions $f(x)$, linear in x and $1/x$. Using the WolframAlpha mathematical engine (ME) we found that the equations hold true for most of the real numbers domain $x \in \mathbb{R}$ such as the algebraic, irrational and transcendental excluding some integer numbers to avoid division by zero in denominators. A further generalization to the complex numbers domain $x \in \mathbb{C}$ and under specific conditions also for hypercomplex numbers such as quaternions was also tested numerically.

Keywords: arctangent, Machin formula, π calculations.

1. Introduction

Most publications to date show that researchers in this endeavor concentrated on developing Machin type parametric formulae rather than functional equations [1-5]. This is quite understandable in view of the fact that the original historical task was to obtain rather formulae for more efficient numerical calculations of π, lacking at the time the current fast computers and more advanced computer algorithms.
The following "Machin formula" was discussed in many papers [3] and has been in use for more than 3 centuries to calculate π with an ever increasing accuracy:

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}) \quad (1)$$

Formula (1) was employed in the calculation of π because it converges relatively faster in comparison to typical infinite Taylor series such as the Gregory-Leibniz equation [1] sequence:

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots \quad (2)$$

This infinite series gives for $x = 1$ the well known infinite series expansion of $\frac{\pi}{4}$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}\dots \quad (3)$$

Many more Machin type formulae like eqn.(1) attracted a lot of interest due to their improved computational converging properties to π. On the one side of these formulae appears usually the number $\pi/4$ though in some cases also other rational multiples of π, while on the other side various combinations of sums of arctangents with integer numbers [4,5].
Most publications of Machin type formulae used in computer searches were given algebraically with specific integer or rational coefficients but without reference to any geometrical interpretation.
In this paper we give geometrical proofs and generalization of the following two simple arctangent formulae published in the literature [3,6-9] which display symmetry and elegance:

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi \quad (4)$$
$$\arctan(1) + \arctan(\tfrac{1}{2}) + \arctan(\tfrac{1}{3}) = \frac{\pi}{2} \quad (5)$$

The following general finite series extension is discussed briefly in [3,10]:

$$\frac{\pi}{4} = \sum_{k=1}^{n} a_k \arctan(\frac{1}{b_k}) \quad (6)$$

where a_k are integers, either positive or negative, and b_k are positive integers. In particular, the investigation presented here is to extend eqns.(4) and (5) for the purpose of finding functions $f_i(x)$ of a single variable x which satisfy the following specific functional form:

$$\arctan f_1(x) + \arctan f_2(x) + \arctan f_3(x) = \frac{\pi}{p} \quad (7)$$

where p is an integer or rational number and finding the largest possible domain for the variable x starting with the positive integers domain $x = 1,2,3,4$ ...
In this paper we make a distinction between the terms "Machin type Formulae" and "Machin type Equations", realixing that the first terminology (Formulae) relates to specific rational parametric numbers expression such as eqns.(1), (4), (5) and (6), while the second terminology (Equations) refers to functions displaying the arctan variable as a function *f(x)* operating on well defined numbers domains such as eqn. (7).

2. π relation to arctangent functions

The $tan(x)$ function, like all periodic functions is not a bijective or so called one to one function. Nevertheless, its inverse i.e., $arctan(x)$ has some advantageous properties. Its domain accepts all the real numbers $x \in \mathbb{R}$ although its range is usually restricted between the principal values $\pm\pi/2$. This allows one to use various types of $arctan(x)$ sums to find new expressions for π. We also know that it is a special function of x which gives simple integer geometric ratios in right angle triangles. Thus for 3 angles α, β and γ, and triangles with sides ratios based on any length units a, which can be integer, rational or even irrational numbers like $\sqrt{2}$ if:

$$\tan(\alpha) = \frac{a}{a}; \quad \tan(\beta) = \frac{2a}{a}; \quad \text{and} \quad \tan(\gamma) = \frac{3a}{a} \quad (8)$$

From (8) one can show quite trivially the following algebraic result:

$$\arctan\left(\frac{a}{a}\right) + \arctan\left(\frac{2a}{a}\right) + \arctan\left(\frac{3a}{a}\right) = \arctan(1) + \arctan(2) + \arctan(3) = \pi \quad (9)$$

The practical outcome using this function is that one can derive some elegant and symmetric equations for expressing π as shown in this paper.

3. The arctangent formula for 3 angles

One straigthforward geometrical demonstration of equation (4) is well presented in [6] while the proof of eqn (5) can be found for instance in [8]. However we reproduce next for clarity, two alternative geometrical version proofs.

We refer thus in Fig.1 to the geometry of the two right angle triangles ABC and ADE drawn onto a Cartesian 2D lattice-grid of unit side length squares. From the angles shown extending from point B of the right angle triangle ABC we obtain the tangents values pertaining to the angles arctan(1), arctan(2) and arctan(3) and from the equilateral right angle triangle ADE we obtain the angles values of arctan(1), arctan(1/2) and arctan(1/3). The internal angle β of triangle ABC is complemented externally by the angles α and γ to $180^0 = \pi$. Since $\alpha = \arctan(1) = 45^0 = \frac{\pi}{4}$ and observing also in the Cartesian lattice that $\beta = \arctan\left(\frac{2\sqrt{2}}{\sqrt{2}}\right) = \arctan(\frac{2}{1}), \gamma = \arctan\left(\frac{3}{1}\right)$ we have the following elegant relation form:

$$\alpha + \beta + \gamma = \pi \quad (10)$$

which proves formula (4), i.e., $\arctan(1) + \arctan(2) + \arctan(3) = \pi$

Similarly, in Fig.1, in the equilateral right angle triangle ADE the sum of the angles $(\theta + \varphi)$ is complemented externally to $\pi/2$ by the angle α:

$$\alpha + \theta + \varphi = \frac{\pi}{2} \quad (11)$$

Where $\theta = arctan(\frac{FE}{ED}) = arctan\left(\frac{\sqrt{2}/2}{\sqrt{2}}\right) = arctan(\frac{1}{2})$, $\varphi = \arctan\left(\frac{BG}{DG}\right)$=$\arctan\left(\frac{1}{3}\right)$ and FE, ED, BG, DG designate segments between corresponding points F,E,D,B,G. Point F divides the segment AE in half.

These last relations prove formula (5), i.e., $\arctan(1) + \arctan(1/2) + \arctan(1/3) = \pi/2$

One may notice also from the two triangles constructions in Fig.1 that the above-mentioned angles complement rather externally of the 2 triangles to π or $\pi/2$ respectively.

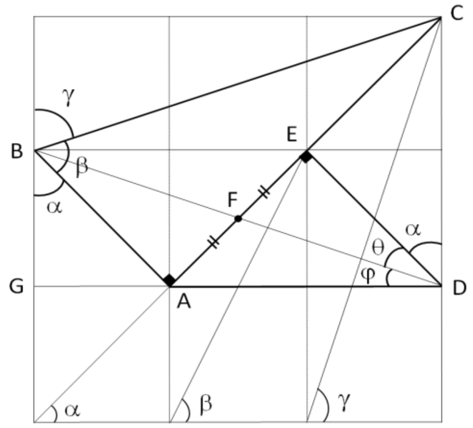

Fig. 1: Diagram of ABC and ADE triangles in a Cartesian lattice of unit sides, showing the angles relations $\alpha + \beta + \gamma = \pi$ and $\alpha + \theta + \varphi = \frac{\pi}{2}$

Similar geometrical procedures are used next to inscribe a series of triangles Δ_k in the Cartesian lattice, where k are positive integers $k = 1,2,3,4\ldots n$, for the purpose of obtaining a series of $arctan(x)$ expressions. The lattice-grid method is employed here also, which tiles the Cartesian 2D plane with squares of unit size onto which we construct the series of "integer indexed triangles" Δ_k with sides magnitudes consisting of either integer numbers or integer multiple magnitudes of $\sqrt{2}$.

4. Arctan identities of type 1

Onto the lattice-grid of unit size squares in the Cartesian plane in Fig. 2 we plot individual right angles from the points P,A,B,C,D,.. and draw triangles with the points S_1, S_2, S_3, S_4,...S_n and origin point O. These series of acute triangles Δ_k drawn onto the Cartesian plane are indexed by the integers k=1,2,3,4…n. Around every point S_k, pertaining to the triangles series Δ_k, the 3 angles are indexed as u_k, v_k, α_k which complement together to π, i.e., $u_k + v_k + \alpha_k = \pi$. We note nevertheless that only the v_k angles are internal in each of the k^{th} triangle and the other two, u_k and α_k are external. In particular, for k =1, the triangle OS_1A, denoted as Δ_1, the three angles summing up to π are given by $u_1 + v_1 + \alpha_1 = \pi$.
From Fig.2 one may observe that for all the continuing series of n triangles we have:

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \ldots. = \alpha_n = \frac{\pi}{4} \tag{12}$$

$$\tan(u_1) = \frac{3}{1};\quad \tan(u_2) = \frac{4}{2};\quad \tan(u_3) = \frac{5}{3};\quad \tan(u_4) = \frac{6}{4}\ldots \tag{13}$$

$$\tan(v_1) = \frac{2\sqrt{2}}{\sqrt{2}} = \frac{2}{1};\quad \tan(v_2) = \frac{3\sqrt{2}}{\sqrt{2}} = \frac{3}{1};\quad \tan(v_3) = \frac{4\sqrt{2}}{\sqrt{2}} = \frac{4}{1};\quad \tan(v_4) = \frac{5\sqrt{2}}{\sqrt{2}} = \frac{5}{1}\ldots \tag{14}$$

Thus, adding geometrically the complementing angles to 180°= π for each of the k^{th} triangle in the series k =1,2,3,4...n, in the Cartesian plane of Fig. 2 we obtain n relations as parametric formulae F_{Sk}:

$$\begin{aligned}
u_1 + v_1 + \alpha_1 = \pi \quad &\Rightarrow\; F_{S1} = \arctan\left(\frac{3}{1}\right) + \arctan\left(\frac{2}{1}\right) + \frac{\pi}{4} = \pi\\
u_2 + v_2 + \alpha_2 = \pi \quad &\Rightarrow\; F_{S2} = \arctan\left(\frac{4}{2}\right) + \arctan\left(\frac{3}{1}\right) + \frac{\pi}{4} = \pi\\
u_3 + v_3 + \alpha_3 = \pi \quad &\Rightarrow\; F_{S3} = \arctan\left(\frac{5}{3}\right) + \arctan\left(\frac{4}{1}\right) + \frac{\pi}{4} = \pi\\
u_4 + v_4 + \alpha_4 = \pi \quad &\Rightarrow\; F_{S4} = \arctan\left(\frac{6}{4}\right) + \arctan\left(\frac{5}{1}\right) + \frac{\pi}{4} = \pi\\
&\vdots\\
u_n + v_n + \alpha_n = \pi \quad &\Rightarrow\; F_{Sn} = \arctan\left(\frac{n+2}{n}\right) + \arctan(n+1) + \frac{\pi}{4} = \pi,
\end{aligned} \tag{15}$$

From the above series of n-formulae we conclude now by induction that for any positive integer variable number x we obtain by generalization also the following integer variable function $F^+(x)$:

$$F^+(x) = \arctan\left(\frac{x+2}{x}\right) + \arctan(x+1) + \arctan(1) = \pi\,, (x \in \mathbb{Z}^+, \forall x > 0) \tag{16}$$

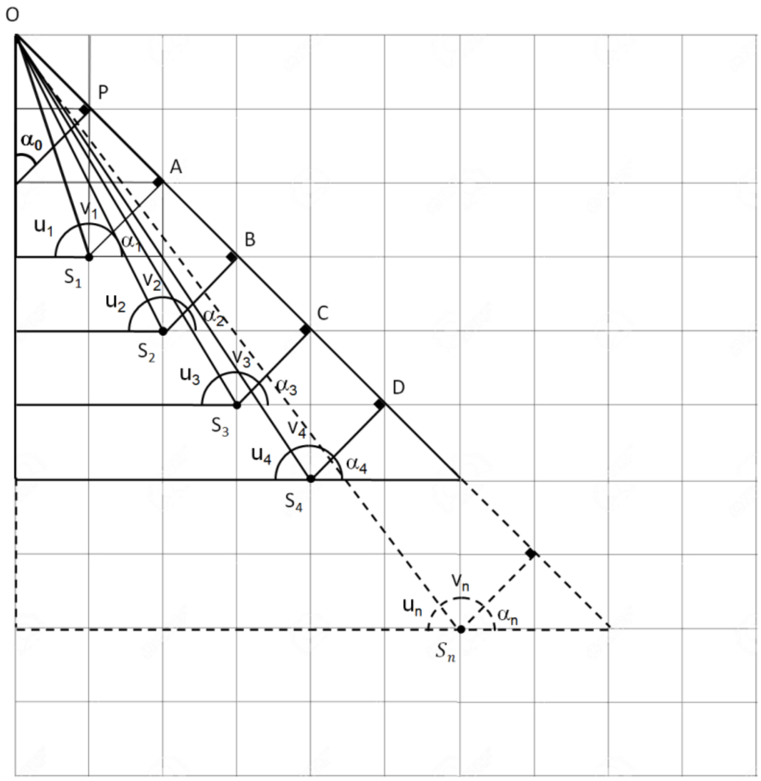

Fig. 2: Diagram of the Δ_k triangles in a Cartesian grid, demonstrating the angles complementarity $u_k + v_k + \alpha_k = \pi$ and their relationships to the functions of arctan detailed in Equations (15).

Then using the mathematical engine numerical platform WolframAlpha (ME) [11] we realize that the $F^+(x)$ identity is also true for any positive real number $x \in \mathbb{R}^+$, thus also rational, irrational and transcendental numbers such as 0.1, π, e, $\sqrt{2}$, π^e, e^π, $\sqrt{2}^{\sqrt{2}}$ and their combinations for which only the last one is given below:

$$\text{For } x = \sqrt{2}^{\sqrt{2}} \Rightarrow \arctan\left(\frac{\sqrt{2}^{\sqrt{2}}+2}{\sqrt{2}^{\sqrt{2}}}\right) + \arctan\left(\sqrt{2}^{\sqrt{2}} + 1\right) + \arctan(1) = \pi$$

This identity or equation (16) is satisfied thus for any positive real number $x \in \mathbb{R}$ with $x > 0$. Testing further the equation (16) using WolframAlpha (ME), we verified it also for the complex numbers domain $x \in \mathbb{C}$ finding that this identity holds true also in the right half Cartesian plane i.e., $x = a + ib$ for any real number $a > 0$ and for any $b \in R$ as shown in the following example for $x = 2 - 3i$:

$$\arctan\left(\frac{2-3i+2}{2-3i}\right) + \arctan(2-3i+1) + \arctan(1) = \frac{\pi}{4} + i\,arctanh\left(\frac{17}{13} + \frac{6}{13}i\right) - \arctan(3+3i) = \pi$$

For $a < 0$, i.e., the real negative numbers domain and for any real number b the proper equation changes to:

$$F^-(\mathrm{x}) = \arctan\left(\frac{\mathrm{x}+2}{\mathrm{x}}\right) + \arctan(\mathrm{x}+1) + \arctan(1) = 0\,,\ (\forall x < 0)$$

Thus the general formal function F(x) for the whole Complex plane numbers domain other than $x = 0$ is:

$$\mathrm{F(x)} = \arctan\left(\frac{\mathrm{x}+2}{\mathrm{x}}\right) + \arctan(\mathrm{x}+1) + \arctan(1) = \begin{cases} F^+(x) = \pi\,, \forall x > 0 \\ F^-(\mathrm{x}) = 0\,, \forall x < 0 \end{cases} \quad (17)$$

The equation (16) i.e., $F^{+}(x)$ was investigated also with WolframAlpha (ME)) for quaternions of the type $(1, i, j, k)$ giving the following result for $x = 1 + i + j + k$:

$$\arctan(\frac{1+i+j+k+2}{1+i+j+k}) + \arctan(1 + i + j + k + 1) + \arctan(1) = \frac{1}{2}\left(\pi + i(\ln(1 - i) - \ln(1 + i))\right) + O\left(\left(\frac{1}{j}\right)^2\right) + \frac{\pi}{4} = \frac{1}{2}\left(\pi + \frac{\pi}{2}\right) + O\left(\left(\frac{1}{j}\right)^2\right) + \frac{\pi}{4} \underset{j\to\infty}{\Longrightarrow} \pi$$

5. Arctan identities of type 2

The geometrical demonstration of formula (5) with the inverse form of the type $1/x$ for positive integers x was already shown above and can be found also in [3]. This equation is generalized now as a functional summing relation as shown next.

In Fig.3 we still use a Cartesian lattice-grid of unit sides squares in which we draw right angle equilateral triangles $\Delta_{\mathbf{q}}$ drawn from the right angles vertices points A,B,C,D,E,F,… designated by the index q i.e., q=A,B,C,D,E,F…. We proceed then to calculate the tangents of the angles β_q, γ_q that complement the angles α_q = 45° to 90°= $\frac{\pi}{2}$ in each triangle $\Delta_{\mathbf{q}}$ from their sides projections onto the ordinate and abscissa axes.

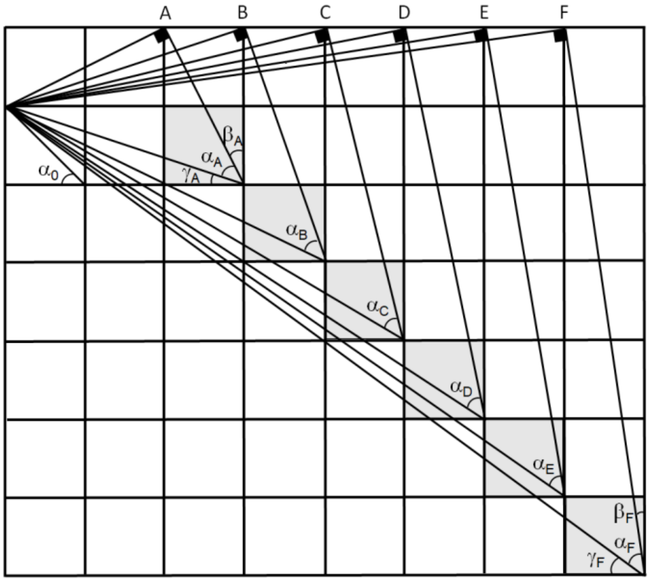

Fig. 3: Diagram of the Δ_q equilateral right angle triangles on the Cartesian grid, demonstrating the angles complementary $\alpha_q + \beta_q + \gamma_q = \frac{\pi}{2}$, for q=A,B,C… and their relationships to the functions of arctan detailed in Equations (21,22).

In this case, we can summ up the angles designated as $\alpha_q, \beta_q, \gamma_q$ in each of the shaded squares of the lattice of Fig. 3 to obtain the following relations:

$$\alpha_0 = \alpha_A = \alpha_B = \alpha_C = \alpha_D = \alpha_E = \alpha_F = \dots = \alpha_W = \frac{\pi}{4} \quad (18)$$

$$tan(\beta_A) = \frac{1}{2};\ tan(\beta_B) = \frac{1}{3};\ tan(\beta_C) = \frac{1}{4};\ tan(\beta_D) = \frac{1}{5};\ tan(\beta_E) = \frac{1}{6};\ tan(\beta_F) = \frac{1}{7} \quad (19)$$

$$tan(\gamma_A) = \frac{1}{3};\ tan(\gamma_B) = \frac{2}{4};\ tan(\gamma_C) = \frac{3}{5};\ tan(\gamma_D) = \frac{4}{6};\ tan(\gamma_E) = \frac{5}{7};\ tan(\gamma_F) = \frac{6}{8} \quad (20)$$

$$\alpha_A + \beta_A + \gamma_A = \frac{\pi}{2} \Rightarrow \arctan(1) + \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{2}$$

$$\alpha_B + \beta_B + \gamma_B = \frac{\pi}{2} \Rightarrow \arctan(1) + \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{2}{4}\right) = \frac{\pi}{2}$$

$$\alpha_C + \beta_C + \gamma_C = \frac{\pi}{2} \Rightarrow \arctan(1) + \arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{3}{5}\right) = \frac{\pi}{2}$$

$$\alpha_D + \beta_D + \gamma_D = \frac{\pi}{2} \Rightarrow \arctan(1) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{4}{6}\right) = \frac{\pi}{2}$$

$$\alpha_E + \beta_E + \gamma_E = \frac{\pi}{2} \quad \Rightarrow \arctan(1) + \arctan\left(\frac{1}{6}\right) + \arctan\left(\frac{5}{7}\right) = \frac{\pi}{2}$$

$$\alpha_F + \beta_F + \gamma_F = \frac{\pi}{2} \quad \Rightarrow \arctan(1) + \arctan\left(\frac{1}{7}\right) + \arctan\left(\frac{6}{8}\right) = \frac{\pi}{2} \quad (21)$$

$$\vdots$$

From eqn.(21) we conclude by induction that for any integer x we obtain here a general function $E(x)$ given by:

$$E(x) = \arctan(1) + \arctan\left(\frac{1}{x+1}\right) + \arctan\left(\frac{x}{x+2}\right) = \frac{\pi}{2} \quad \text{with } x \neq -1 \text{ and } x \neq -2 \quad (22)$$

Simple substitutions in equation $E(x)$ prove that it holds true also for 0 and all negative integer numbers x except for the numbers $x \neq -1$ and $x \neq -2$ to avoid division by zero. Below is shown an example for the negative number $x = -8$:

$$arctan(1) + arctan(\frac{1}{-8+1}) + arctan(\frac{-8}{-8+2}) = \frac{\pi}{2}$$

One may observe that the equation $E(x)$ has a larger validity domain including all integer numbers if compared to equation $F^+(x)$ which is true only for positive numbers i.e., $x > 0$.
We tested $E(x)$ equation validity beyond the integers numbers domain, using again the WolframAlpha (ME) platform realizing that it holds true also for all types of real numbers $x \in \mathbb{R}$ such as rational, irrational, transcendental numbers and even irrational powers combinations such as: $0.1, \pi, e, \pi^e, e^\pi, \sqrt{2}^{\sqrt{2}}$.
Further testing with WolframAlpha (ME) one can show that these Machin type equations $E(x)$ are satisfied also in the complex numbers domain i.e., for $x = a + bi$ for any real a and b, thus:

$$E(a+bi) = arctan(1) + arctan\left(\frac{1}{a+bi+1}\right) + arctan\left(\frac{a+bi}{a+bi+2}\right) = \frac{\pi}{2}$$

Testing equation (22) further for the quaternions domain we obtained the following result for $x = 1 + i + j + k$:

$$arctan(1) + arctan\left(\frac{1}{1+i+j+k+1}\right) + arctan\left(\frac{1+i+j+k}{1+i+j+k+2}\right) == \frac{\pi}{4} + \frac{1}{2} i(\ln(1-i) - \ln(1+i)) + O\left(\left(\frac{1}{j}\right)^2\right) \underset{j\to\infty}{\Longrightarrow} \frac{\pi}{2}$$

This generalization states that the single real variable function $E(x)$, $x \in \mathbb{R}$ is satisfied also for the 2D complex numbers domain $x \in \mathbb{C}$, and maybe even by higher hypercomplex numbers such as quaternions, etc., though for them only for specific limits.

6. Conclusions

This investigation followed the observation of the elegance and symmetry of the two original basic Machin type formulae (4) and (5). We suggested here a generalization of these parametric formulae using the functions summing approach, $\pi/p = \sum_{k=1}^{n} arctanf_k(x)$ with integer or rational p, using two variant triangles geometrical constructions obtaining the first type identity i.e., eqn.(16):

C1. $\arctan\left(\frac{x+2}{x}\right) + \arctan(x+1) + \arctan(1) = \pi$ for any real positive $x > 0$

And the second type identity i.e., eq. (22):

C2. $$arctan(1) + arctan\left(\frac{1}{x+1}\right) + arctan\left(\frac{x}{x+2}\right) = \frac{\pi}{2},$$

Which is true for any real positive and negative real number including $x = 0$ and excluding only $x \neq -1$ and $x \neq -2$.

The former equations validity for the positive integers was found true not only for the real numbers domain but also for the whole Complex numbers domain under some specific exemptions. This validity was tested for typical numbers using the WolframAlpha mathematical engine platform. The systematic geometrical approach adds an intuitive geometrical feature to Machin type equations derived previously only by algebraic methods.

References

[1] N. Gawronska, D. Slota, R. Witula, A. Zielonka, Some Generalizations of Gregory's Power Series and their Applications, *Journal of Applied Mathematics and Computational Mechanics,* 12(3),79-91, 2013.

[2] S. M. Abrarov et.al, Unconditional applicability of the Lehmer's measure to the two-term Machin-like formula for pi, General Mathematics, 2020.

[3] Y. Nishiyama, Machin's Formula and Pi, *International Journal of Pure and Applied Mathematics,* 82(3), 421-430, 2013.

[4] S. M. Abrarov, B. M. Quine, Efficient computation of pi by the Newton–Raphson iteration and a two-term Machin-like formula, International Journal of Mathematics and Computer Science, 13(2), 157–169, 2018.

[5] Michael Wetherfield, The Enhancement of Machin's Formula by Todd's Process, *The Mathematical Gazette, 80*(488), 333-344, 1996.

[6] M. W. Ecker, An Arctangent Triangle: Puzzling over Arctan1 + Arctan2 + Arctan3 = [pi], *Mathematics and Computer Education*, 40(2), 24-127, 2006.

[7] I. Tweddle, John Machin and Robert Simson on inverse-tangent series for π, Archive for History of Exact Sciences, 42, 1-14, 1991.

[8] S. Breurer, J. Gal-EzerR and G. Zwas, Microcomputer laboratories in mathematics education, *Computers Math. Applic.* 19(3),13-34, 1990.

[9] K. Zelator, The Diophantine Equation arctan(1/x)+arctan(m/y)=arctan(1/k), arXiv:1203.6380 [math.GM], 2012.

[10] S. M. Abrarov, B. M. Quine, An iteration procedure for a two-term Machin-like formula for pi with small Lehmer's measure, arXiv:1706.08835 [math.GM], 2017.

[11] Wolfram Mathematical Engine (ME), https://www.wolfram.com/mathematica/

[12] Kinko Tsuji et.al., Spirals and Vortices: In Culture, Nature, and Science, Springer Nature Switzerland AG, 2019.

[13] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1972.

[14] R. Witula, E.Hetmaniok, D. Slota, Generalized Gregory's series, Applied Mathematics and Computation, 237,203–216, 2014.

[15] J.W. Wrench Jr., On the derivation of arctangent equalities, American Mathematical Monthly, 45, 108-109, 1938.

[16] P. Loya, Amazing and Aesthetic Aspects of Analysis, Springer, New York, USA, 323-325, 2017.