Number – the Basic Concept of Mathematics

Voytsekhovsky Vladimir
vvvaaa1951@gmail.com
vvv1951@mail.ru
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Abstract

The article briefly considers the approaches to the definition of the concept of number by the scientists of Ancient Greece, Aristotle, Frege, Peano, Russell, Dedekind and modern science. The author of this paper proposes basic concept of "***tsel***" to define the concepts of "*unit*" and "*natural number*", and it is shown that the participation of a reasonable subject is necessary in the formation of the concepts of "unit" and "number". Also, the properties of fractional (rational) numbers are considered; it is shown that the reason for the appearance of *infinite rational numbers* is the use of a specific number system, for example, decimal and the resulting incommensurability of some natural numbers, which manifests itself in the fact that it is impossible, for example, to answer the question: "How many numbers *a* (for example, the number *a*=7) is contained in the number *b* (for example, the number *b*=10). The properties of irrational numbers are considered.

Natural numbers

The concept of a natural number (1, 2, 3 ...) is intuitively available to both people and other rational beings. However, when we try to define a number (what is a number at all?), we run into difficulties. These difficulties are due to the fact that definitions have to be given with the help of such basic concepts that are understandable without explanation, and themselves should not and cannot require a definition (after all, something must eventually be accepted without explanation, as a matter of course, understandable to everyone without disputes, definitions).

The definition of "number" was of interest for many philosophers.

Scientists of ancient Greece shared the concepts of Unit (Monas) and Number (arithmos). Greek mathematicians tend to regard a number (arithmos) as a set of units. Their concept includes, according to Mendell [2019, 27-33], the following representations:

1. The number is built from some countable unit (Mona).
2. Numbers are members of the series: 2, 3... Where 1 is understood as the "beginning" (arche) of the number or as the smallest number.
3. Units are comparable if they can be counted together (for example, ten cows in a field). Units are incomparable if it is conceptually impossible to count them together.
4. Aristotle's discussion of time also gives us some insight into the problem of unity. What gives the five black cats a unity is that they can be treated as a unity. Hence, for Aristotle it follows that without the mind there can be no number. Nothing is countable if there is no counter.

It is known the work of Russell [1919, 11-15], in which he writes:

"The question "What is a number?" is one which has often been asked but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his Grundlagen der Arithmetik… It is clear that number is a way of bringing together certain collections, namely, those that have a given number of terms…" At the end, Russell concludes: *"In actual fact, it is simpler logically to find out whether two collections have the same number of terms than it is to define what that number is."*

In addition, Russell investigates the Peano's work on this question [B. Russell 1919, 5-6]:

"The three primitive ideas in Peano's arithmetic are: 0, number, successor.
By "successor", he means the next number in the natural order. That is to say, the successor of 0 is I, the successor of 1 is 2, and so on.

By "number", he means, in this connection, the class of natural numbers and formulates five postulates:

1. 0 is a number.

2. The successor of any number is a number.

3. No two numbers have the same successor.

4. 0 is not the successor of any number.

5. Any property, which belongs to 0, and to the successor of every number, which has the property, belongs to all numbers. "

However, Peano and Russel also do not give an answer to the question "What is a number?"

In the work [Vinogradova E.P. 2014, 186-187] a natural number is defined as *"a measure of the length of a segment, and this number shows how many unit segments the measured segment consists of. Similarly, the concept of a natural number can be introduced in connection with the measurement of other quantities (the areas of figures, the masses of bodies, etc.), and everywhere a natural number as a measure of a quantity will show how many times a unit of measurement is contained in the object being measured."*

Here is the definition of number given by philosophy and mathematics now:
"*Number is one of the most important concepts in mathematics. It is used to describe quantitative characteristics, for comparisons, numbering of objects and their parts"*. Unfortunately, this definition cannot be considered as a complete and acceptable one, because we cannot use the concepts of "quantity" and "numbering" when defining the concept of number – they, in turn, require their definition through the concept of "number". Only concepts that do not raise questions, concepts that are basic, self-evident, axiomatic, can be considered as a complete definition. Let us try to define the concept of "natural number".

Let us introduce the new basic concept of "**tsel**":
t*sel* - *a subject, object, event, thought, ... - everything that is considered or perceived by thinking being as something integral, that is, separate from another, and which, when divided into parts, can lose its main features that are important for the thinking being*. (In English, for the word "***TSEL***" we can offer the abbreviation decoding as "by **T**hinking **S**ubject **E**stimated **L**ike integer object".)

The concept of "*tsel*" is close to the English concept of "entity", but differs from the latter. "Entity" focuses on the existence of objects or their combination into a new whole (https://dictionary.cambridge.org/dictionary/english/entity), *something that exists apart from other things, having its own independent existence*, for example*:*

The museums work closely together, but are separate legal entities.
He regarded the north of the country as a separate cultural entity.
Church and empire were fused in a single entity".

(https://www.merriam-webster.com/dictionary/entity)
: independent, separate, or self-contained existence
: the existence of a thing as contrasted with its attributes
: something that has separate and distinct existence and objective or conceptual reality
: an organization (such as a business or governmental unit) that has an identity separate from those of its members.

https://www.britannica.com/dictionary/entity:
something that exists by itself: something that is separate from other things:;
one division of the company was broken off as a separate entity;
a business/commercial/corporate entity;
government/political/legal entities;
distinct/independent entities).

The concept of "*entity*" coincides with the concept of "*tsel*" in relation to objects, but it is not used in relation to events, for example, such as "lightning strike", "bite", "cry", "sneeze", etc. The concept of "tsel" includes events, and, in addition, clearly indicates the obligatory presence of reason.

Examples of *tsels*: a person, a tree, a leaf of a tree, a match, a box of matches, a bag of grain, half a bag of grain, one grain, a stone, a meteorite fall, an axiom, a decree (order), a flash in the sun, a thought, a cloud, a sneeze, a wave, etc. The concept of "*tsel*" has a dual character: on the one hand, it is objective, independent of the individual mind; on the other hand, it is created and determined by the mind. Here are some examples of tsels that illustrate this duality:

1) Imagine that there is a bundle of logs; each log can be considered as an integral object, that is, a tsel. However, if we split a thick log, then we will get another log, one more *tsel*. The bundle itself, under some circumstances, can also be considered as *tsel*.

2) Imagine that we were attacked by wolves; the number of wolves is of fundamental importance for us for survival: one wolf, or two, or three... If we imagine that wolves attacked a bear, then this is also fundamental for a bear, and it distinguishes one wolf from two, three ... The wolves in this example can be seen as *tsels*. Wolves differ from each other in size, color, etc., but we neglect these differences, since we consider them secondary.

3) In ancient times, the armed forces consisted from a small number of warriors who were *tsels*; modern armed forces number millions of people, and military leaders operate armies, divisions - for them, not warriors but armies and divisions are *tsels*.

In the absence of mind, there is no concept of tsel either. For example, if there are two trees, but there is no mind that would understand this, then these two trees become just a set of material particles.

It is interesting that in nature there are no two identical *tsels* altogether, just as there are no, for example, two identical people, animals, trees, grasses, plant leaves, clouds, matches,... Whether atoms and molecules are the same, it is not known; most likely, they are not - if only by reason that they are in a different environment, and the environment, as we know, affects the neighbors, in this case, energy levels, etc.

However, any *tsel* has something common (the same) with any other *tsel*; we will call this general characteristic "one", or “unit”, or "number 1".

A unit (number 1) *is a characteristic of tsel,* <u>*the same for any tsel*</u>.

If tsel is added to tsel, then their characteristic is the number 2=1+1; adding one more tsel we’ll get number 3=2+1=1+1+1 and so on. Thus, we get a series of natural numbers - the universal characteristics of tsels. ***Natural numbers*** *are universal characteristics of any tsels regardless to the individual characteristics of the latter.*

Zero

The meaning of zero ("0") is the absence of *tsel*. In positional number systems, "0" means the absence of a natural number in the corresponding place.

Whole Numbers

Whole numbers include integers, zero, and integers with a "-" ("minus") sign. The "-" sign before an integer has the following meanings:

1) Decrease in quantity by this integer;
2) Change in the direction of movement or reference to the opposite.

Fractional Numbers (Fractions, Rational Numbers)

Fractional numbers involve dividing the whole into parts (for example, a sack of grain is divided into equal or unequal parts). Equal parts in reality cannot be exactly same, but, with an accuracy that suits us, they **are considered** the same and therefore can become *tsel*s. Fractions are denoted as $\frac{a}{b}$, where *b* is an integer that divides the whole (divisor), that is, $\frac{1}{b}$ means dividing the *tsel* ("1") into *b* parts; "*a*" is an integer number of parts ($\frac{1}{b}$). Rational numbers include in themselves as a special case integers.

It may seem that the division of the whole, that is, the "1", into parts is in some cases possible with absolute precision (e.g.,$\frac{1}{2} = 0{,}5$, $\frac{1}{4} = 0{,}25$, $\frac{1}{5} = 0{,}2, etc.$) and, in some cases, approximately (e.g. $\frac{1}{3} = 0{,}333...or\ 0{,}3\,(3)$, $\frac{1}{6}$ =0,16666… or 0,16(6),
$\frac{1}{7} = 0{,}14285714285711428571...\ or\ 0{,}1428571\,(1428571)$, $-$ because it leads to an infinite sequence of numbers. In fact, infinity appears here because we are actually asking, "How many tenths are contained in $\frac{1}{3}$, or in $\frac{1}{6}$, or in $\frac{1}{7}$?». The point is that we use the decimal number system. However, it looks another if we use, for example, the Nine-Number System with the digits 1, 2, 3, 4, 5, 6, 7, 8 and then $10_9 = 9_{10}$. In the nine-number system there is no digit 9, so we write it down as 1 in the "nines" digit - 10 (further we will go to the numbers $11_9 = 10_{10}$, $12_9 = 11_{10}$, … $18_9 = 17_{10}$, $20_9 = 18_{10}$, $21_9 = 19_{10}$, … $30_9 = 27_{10}$, …).

And so, in the nine-number system, the fraction $\frac{1}{3} = 0{,}3_9$, $\frac{1}{6} = 0{,}15_9$, $\frac{1}{9} = 0{,}1_9$. Actually, infinite rational fractions don't exist; they appear when you try to ask, "How many fractions are contained in another fraction?", but that is like asking, for example: "How many '3' numbers are contained in the number 10"? However, these numbers are incommensurable, so the answer will be, "Go ... to infinity." What is the question - such is the answer.

In practice, there is little difference, other than convenience and speed, between dividing a whole ("1") by 2 or by 3 or any other number of parts. The accuracy (or error) of division does not depend on the divider, but only on the accuracy of the measuring device.

Irrational Numbers

An irrational number is a real number that is not rational, that is, it cannot be represented as an integer or ordinary fraction. An irrational number can be represented as an infinite non-periodic decimal fraction.

As an example of an irrational number, let us consider a right triangle with legs equal to one (1). The hypotenuse of this triangle is $\sqrt{1^2 + 1^2} = \sqrt{2}$ = 1, 41421356237… We do not know this number (it is hidden in infinity), although it can be calculated with any practical accuracy. We obtain the situation: if we take the lengths of the legs as 1, then the exact length of the hypotenuse is unknown to us (we denote this unknown number as $\sqrt{2}$), but if we take as 1 the length of the hypotenuse, then we do not know the lengths of the legs (they are named to be equal to $1/\sqrt{2}$).

The reason for the emergence of irrational numbers is that most integer and rational numbers cannot be obtained by raising to a power (multiplying by itself) any integer or rational number. An irrational number can be calculated with any accuracy necessary for practice, but its exact value cannot be expressed by a rational number; it is "hidden" in infinity.

Not all right triangles have incommensurate sides. There is an infinite number of right triangles in which both legs and hypotenuse have an integer value: so, for any integer n = 1, 2, 3, ... there is a right triangle with legs (2n+1) and 2n×(n+1) and the hypotenuse 2n×(n+1)+1, whose sides are all integers. The smallest and most famous among them is obtained with n=1: this is a triangle with legs 3 and 4 and hypotenuse 5; this fact is used by builders to check for the presence of a right angle. The next such triangle is obtained with n = 2: its legs are 5 and 12, and the hypotenuse is 13. There are other right triangles with integer sides that are not obtained by these relations.

It can be proved that no one irrational number, no matter how far we calculate its value, can have, starting from some position, any digit as a period (that is, it cannot have (1), (2), (3), (4), (5), (6), (7), (8), or (9) in a period).

Every irrational number between 0 and 1 can be represented by an infinite row of fractions.

Real Numbers

Real numbers combine sets of rational and irrational numbers.

Complex Numbers

Complex numbers - numbers of the form $a + bi$, where a, b - real numbers, i - imaginary unit, that is the number for which the equality is satisfied: $i^2 = -1$.

Historically, complex numbers were introduced when calculating the roots of algebraic equations in which, in addition to real roots, roots of the form $(a + b\sqrt{-1})$ also appeared. An example of such equations is $x^2 + 1 = 0$.

Later, when complex numbers were interpreted geometrically as points on the plane, the imaginary unit received a clear meaning.

If the meaning of a negative number on the number axis means a reversal of direction relative to the original point 0 (the number -x is symmetrical to the number x relative to zero), then the meaning of the imaginary unit on the number plane consists in the rotation of a point by 90° counterclockwise (on agreement) around the center of coordinates of axes X and Y. The number x on the real number axis X, being multiplied by the imaginary unit i (that is, number xi), gets to the axis Y with coordinate $y = x$. If we multiply this number xi by i again, we will rotate the point 90° counterclockwise around the center of coordinates of axes X and Y once more, and the point will be again on the axis X, only in the negative region symmetrically to the number x relative to the center of coordinates; we obtain that two rotations gave us $x \cdot i \cdot i = xi^2 = -x$, e.g., $i^2 = -1$.

Gauss argued that "*if the magnitudes 1, -1 u* $\sqrt{-1}$ *were called, respectively, not as positive, negative and imaginary units, but as direct, inverse and incidental, then people would not have the impression that there is some dark mystery associated with these numbers*".

The complex numbers $a + bi$ on the plane correspond to points with coordinates $x = a$ and $y = b$.

The concept of the complex number has proven to be very useful for mathematics, physics, and many branches of technology. It allowed to solve many problems more easily or even to solve problems that could not be solved before. A further development of the concept of the complex number was the quaternion, which, in contrast to the field of complex numbers for the plane, contains three imaginary units, traditionally denoted by i, j and k for three- or four-dimensional space. Quaternions is defined as the sum of

$q=a + bi + cj + dk$,

Where a, b, c, d are real numbers;

i, j, k are imaginary units with the following property $i^2=j^2=k^2=ijk=-1$, and the result of their pairwise product depends on the order of succession (it is not commutative): $ij=k$, but $ji=-k$.

Transcendental Numbers

Real and complex numbers that are not algebraic (An algebraic number is a number that is a root of a non-zero polynomial in one variable with integer (or, equivalently, rational) coefficients), such as π and e, are called transcendental numbers.

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