the general science journal

Generalized Calabi Triangle Problem

Dragan N. Turanyanin*

Belgrade, SERBIA

The classical Calabi triangle problem identifies a unique non-equilateral isosceles triangle in which a square inscribed on the base and two equal squares inscribed on the slanted sides have equal areas. Here, we present a systematic generalization of this problem by introducing additional symmetric square configurations, deriving analytic relations among their areas, and exploring conditions of equality.

1. Introduction

In any isosceles triangle, several geometrically distinct methods allow a square to be inscribed. The Calabi[1] triangle problem [1][2] focuses on the equality of areas among a base-rest square and two lateral (slanted-side) squares (details in Appendix A). Here we extend this framework by considering additional configurations — the so-called 'diamond' squares — defined by symmetry about the triangle's altitude (an example of the setup in [3]; details in Appendix B). This generalization provides a unified algebraic view of several classical and new equal-area conditions. Two triangles emerge with some unexpected regular polygons and polyhedron connections (see Addendum).

2. Square Configurations and Scaled Area Formulas

Let an isosceles triangle have base width w and height h, and define the ratio q = h/w. We consider four non-congruent square types inscribed in or upon this triangle:

1. **Base-rest square** (S_b): one side lies along the base. By similar triangles, the square's side length is

$$s_b = \frac{w \cdot h}{w + h}.$$

   Scaled area[2]:

$$A_b = \frac{q^2}{(q+1)^2}.$$

2. **Lateral squares** (S_1, S_2): each (S_s) rests on one of the equal slanted sides. The side length can be expressed as

$$s_s = \frac{w \cdot q\sqrt{1 + 1/4q^2}}{1 + q + 1/4q}.$$

   Scaled area:

$$A_s = \frac{q^2(16q^2+4)}{[4q(q+1)+1]^2} = \frac{4q^2(4q^2+1)}{(2q+1)^4}.$$

* Independent Researcher; e-mail: turanyanin@yahoo.com

1 Eugenio Calabi

2 Divided by w^2

3. **Diamond squares** (S_d): symmetric about the altitude, with one vertex at the midpoint of the base. Two variants exist (details in Appendix B):
   - S_m (acute case): side

$$s_m = \frac{w \cdot q\sqrt{2}}{1+2q},$$

area

$$A_m = \frac{q^2}{(q+1)^2}.$$

   - S_h (obtuse case): side

$$s_h = \frac{h\sqrt{2}}{2},$$

area

$$A_h = \frac{q^2}{2}.$$

3. Equal-Area Conditions

Setting pairs of these areas equal[3] yields specific ratios q = h/w corresponding to distinct isosceles triangles, i.e. related apex angles C = 2arctan(1/2q):

- $A_m = A_b$ → q = √2/2 ≈ 0.7071 → apex angle[4] C ≈ 70.53° (see Figure 1).
- $A_h = A_b$ → q = √2 − 1 ≈ 0.4142 → apex angle C ≈ 100.72° (see Figure 2).
- $A_b = A_s$ → q = √3/2 ≈ 0.8660 → apex angle C = 60°.
- $A_d = A_s$ → q = 1/2 → apex angle C = 90°.

Each equality defines a unique isosceles triangle. The second case (obtuse diamond–base equality) produces an apex angle within about one degree of the classical Calabi triangle (≈101.74°), suggesting a close geometric relationship.

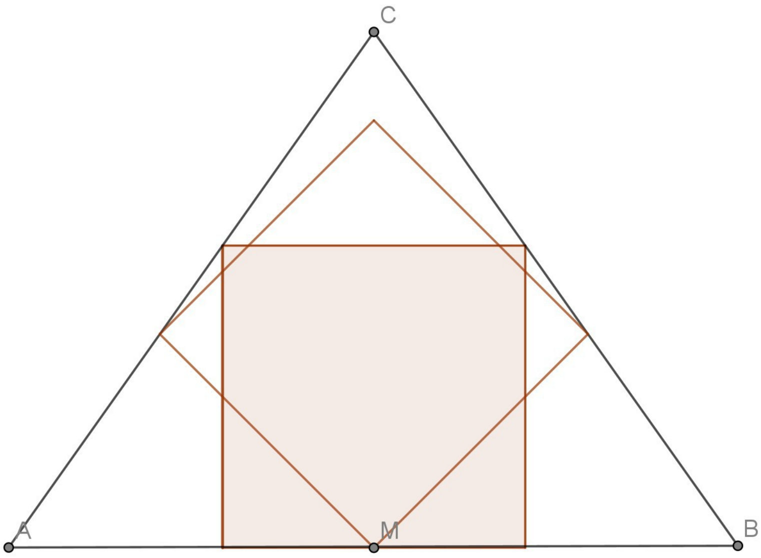

Figure 1: *Acute (base : diamond–base) equality*[5]

3 We left interested reader to conduct the algebra in details.
4 Obviously, base angle A = B = (180 - C)/2
5 Figures made by GeoGebra

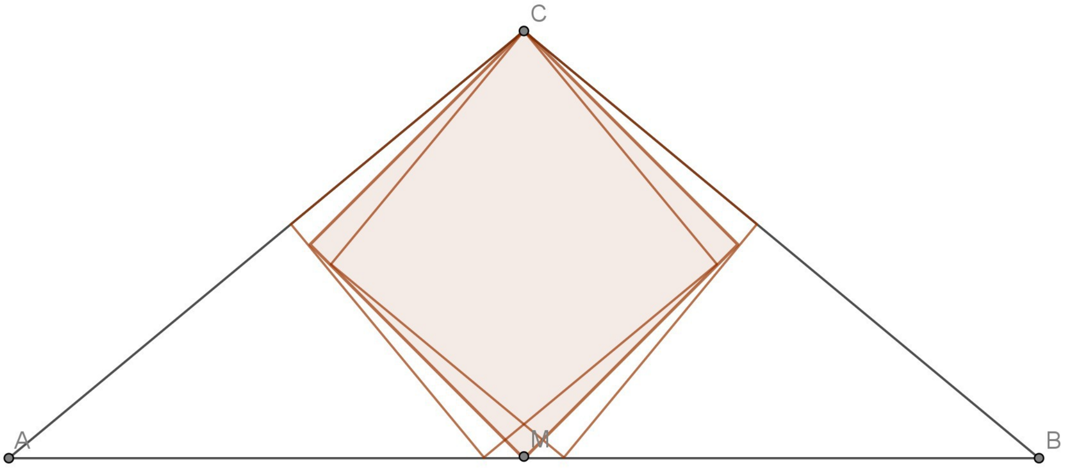

Figure 2: *Obtuse (diamond–base : slant) equality*

4. Ultimate Generalization

The ultimate generalization asks whether a single triangle exists where all non-congruent square types share the same area, i.e. $A_b = A_s = A_d$. Solving these equations reveals that the pairwise equalities occur at distinct ratios ($q = \sqrt{2}/2, \sqrt{2} - 1, \sqrt{3}/2$), so no non-degenerate isosceles triangle satisfies all simultaneously. Only the degenerate limit $h = 0$ makes all areas equal trivially.

5. Discussion

This analysis provides a coherent algebraic framework uniting the Calabi triangle and several new equal-area configurations. The distinct ratios defining equality between different square types trace a continuous family of isosceles triangles ranging from acute to obtuse. The close numerical proximity between the classical Calabi apex and the $S_h = S_b$ equality invites further investigation into possible underlying geometric symmetries. Future work may include:

- exploring transitions between configurations
- extensions to scalene triangles
- geometric loci corresponding to equal-area constraints among multiple inscribed polygons

Acknowledgment

AI disclosure: Besides some symbolic derivations, here GPT-5 (by OpenAI, free version) is partly used as text formatting tool; main both algebraic and conceptual frameworks as well as interpretations are the author's own.

References

[1] Calabi Triangle, *Wikipedia*

[2] Weisstein, Eric W., "Calabi's Triangle", From *MathWorld*, AWolfram Resource.

[3] Turanyanin, Dragan N. Λ-Triangle, *GSJ*, (2025)

Appendices

Appendix A: Derivation of the Calabi Triangle Condition

The Calabi triangle [1] is a unique non-equilateral isosceles triangle that allows three equal maximal squares to be inscribed—one along its base and two along its slanted sides. This appendix outlines a step-by-step geometric derivation (often absent from referent literature) of the algebraic conditions that define the triangle.

Coordinates and notation

Consider an isosceles triangle ABC with base AB of length a and equal sides AC = BC = b. Place the base symmetrically on the x-axis with coordinates A(-w/2, 0), B(w/2, 0), and C(0, h). The altitude h satisfies $b^2 = (w/2)^2 + h^2$. Define the ratio x = b/w.

Square with side on the base

A maximal inscribed square with its base along AB and centered on the base has its top side touching the equal sides AC and BC. By similar triangles, one finds its side length:

$$s_b = (w \cdot h) / (w + h).$$

Squares with sides on the slanted sides

A square can also be inscribed with one of its sides lying on a slanted side (say AC). Using coordinate geometry and the direction vectors along AC and the inward normal, the square's side length can be expressed as:

$$s_s / w = x / (1 + q + 1/(4q)),$$

where $q = h/w = \sqrt{x^2 - 1/4}$.

Equality of inscribed squares

The Calabi condition requires that all three inscribed squares be equal in size. Substituting the two expressions gives:

$$q / (1 + q) = x / (1 + q + 1/(4q)).$$

After algebraic simplification, using $q^2 = x^2 - 1/4$, this reduces to the cubic equation:

$$2x^3 - 2x^2 - 3x + 2 = 0.$$

Result and interpretation

The equation has two meaningful real solutions: x = 1 (the equilateral triangle) and x ≈ 1.551387 (the Calabi triangle). The corresponding angles are approximately 39.13°, 39.13°, and 101.74° [2]. Thus, the Calabi triangle is the only non-equilateral isosceles triangle for which the largest inscribed square may be positioned in three distinct yet equal ways.

Appendix B: Midpoint-Vertex Square in an Isosceles Triangle

This outline presents a brief analysis of the 'midpoint–vertex' square (the so-called 'diamond' configuration, here denoted S_d) that can be inscribed in any isosceles triangle. It complements the classical constructions such as those in the Calabi-type problem (denoted S_b, S_s). The focus here is on the acute-case (denoted S_m) of the geometric setup and related derivations. The obtuse-case (denoted S_h) follows straightforwardly[6].

Geometric setup

Consider an isosceles triangle ABC with base AB of length a, midpoint M(0, 0), and apex C(0, h). Let the equal sides be AC = BC = b. The slope of each slant is m = 2h/w, and we define q = h/w. We aim to inscribe a square S_m such that:

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6 $s_h = h\sqrt{2}/2$, where s_h is square side of the obtuse-case setup (see Figure 2)

1. One of its vertices lies at M, the midpoint of the base; 2. Two of its vertices touch the slanted sides AC and BC; 3. The fourth vertex lies on the altitude (the symmetry axis of the triangle).

Orientation and coordinates

By symmetry, the square must be rotated by 45°. Placing M at the origin (0, 0) and letting $t = s_m$ be the square's side, its vertices can be written as:

$$V_0 = (0, 0),\ V_1 = (t/\sqrt{2}, t/\sqrt{2}),\ V_3 = (-t/\sqrt{2}, t/\sqrt{2}),\ V_2 = (0, \sqrt{2}\cdot t).$$

The top vertex V_2 lies automatically on the altitude x = 0, while V_1 and V_3 must lie on the slanted sides BC and AC, respectively.

Incidence condition and derivation

Equation of the right slanted side BC is $y = m(w/2 - x)$. Substituting $V_1(t/\sqrt{2}, t/\sqrt{2})$ gives:

$$t/\sqrt{2} = m(w/2 - t/\sqrt{2}).$$

Rearranging terms and solving for t yields:

$$t = (\sqrt{2}\cdot m\cdot w) / [2(1 + m)].$$

Substituting $m = 2h/w = 2q$ gives the scaled form:

$$t/w = (\sqrt{2}\cdot q) / (1 + 2q).$$

Summary

The 'midpoint–vertex' square S_d is a valid inscribed square configuration for all isosceles triangles. The configuration remains distinct from the classical Calabi-related squares (which involve equal side or base contacts), and it enriches the overall taxonomy of square-in-triangle relationships.

Addendum: Regular Polygons and Polyhedron Connections

I Apex angle≈70.53° Triangle

This triangle acts as a geometric "bridge" between the 2D symmetry of a square and the 3D symmetry of a tetrahedron (which is composed of equilateral triangles). Here is how this specific triangle—with its ratio $q=\sqrt{2}/2$—mediates those two worlds:

The Connection to the Square

The ratio $q=\sqrt{2}/2$ is exactly half the diagonal of a unit square. Taking a square with side w, its diagonal is $w\sqrt{2}$. The triangle's height h is exactly **half** of that diagonal ($h=w\sqrt{2}/2$).

This means if one imagines a square lying flat, and one "folds" its corners up to meet at a single point above the center to form a pyramid, the height of those triangular faces matches the specific q ratio.

The Connection to the Equilateral Triangle

Here, let us notice the legs of our triangle are $s=w\sqrt{3}/2$. This looks familiar: In a standard **equilateral triangle** with side w, the height is exactly $w\sqrt{3}/2$.

So, the triangle is a hybrid:

- **Its Base (*w*):** Shared with a square.
- **Its Legs (*s*):** Equal to the height of an equilateral triangle of the same base.

The "Mediator" in 3D Space

The most famous place this specific triangle "lives" is inside a r**egular tetrahedron**[7]. If we drop a perpendicular line from one vertex of a tetrahedron to the opposite face, and then draw a line from that point to another vertex, the constructed cross-section is exactly the triangle we described.

It is the specific geometry required to transition from a 2D flat plane into the most basic perfectly symmetrical 3D solid. It "tilts" the symmetry of the square just enough to meet the requirements of the equilateral faces.

II Apex angle≈100.72° Triangle

The triangle is governed by familiar √2 connecting it with inner square's, octagon's and alike symmetries.

Geometric Connections

The ratio $q=\sqrt{2}-1$ is the **silver ratio**'s (δ_S)[8] fractional part and specifically the value of $tan(22.5°)$, i. e. $tan(\pi/8)$. This leads to several interesting connections:

- **Silver Ratio Relationship**: The value $\sqrt{2}-1$ is the silver ratio's reciprocal, i. e. $q=1/\delta_S$. In geometry, this relation often appears in regular octagons.
- **The Octagon Connection**: If we take a regular octagon, the ratio of the distance from the center to a side (apothem) versus half the side length is exactly δ_S. Conversely, q relates to the angles formed by segments within the octagon.
- **Trigonometric Identity**: The base angle (β) of this triangle is exactly $arctan[2(\sqrt{2}-1)]$.

While not a standard "named" triangle like the Golden Triangle, it is a specific "Silver" variant where the height-to-base proportion is defined by the half-angle tangent of 45°.

March, 2026

7 at wikipedia.org/wiki/Regular_tetrahedron
8 $\delta_S=1+\sqrt{2}$; at wikipedia.org/wiki/Silver_ratio