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Numerical Sets in Finitist Mathematics

On the 'rationality' of exact rational numbers
and the 'irrationality' of irrational numbers

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Abstract. This article discusses the mathematical reality of numerical sets once the inconsistency of the actual infinity has been demonstrated (see below). Under these finitist conditions, only three numerical sets can consistently exist: the set of natural numbers, the set of integer numbers, and the set of exact rational numbers; the three of them potentially infinite. Since all irrational numbers have an actual infinite (and then inconsistent) number of decimal places, they are all, as one the meaning of its name indicates, irrational in the sense of inconsistent numbers. This formal "irrationality" of irrational numbers implies that the set of real numbers that contains them all cannot be a consistent set.

Keywords: actual infinity, potential infinity, numerical sets, densely ordered sets, natural numbers, integer numbers, rational numbers, irrational numbers, real numbers.

1. Introduction

This article assumes the inconsistency of the actual infinity, formally demonstrated by the author, among many other works, here [4] and here [3]. An immediate consequence of this inconsistency is that all sets, numerical or otherwise, must be finite or potentially infinite (the following section defines both types of sets). The same applies to the number of decimal places in all numbers that have them, which in fact implies that all numbers are natural, integer, or exact rational numbers. A truly significant consequence of the inconsistency of the actual infinity on the numerical sets. This article briefly analyzes them in the following five sections.

It sounds like an ironic warning that, as we will see (and as one the meaning of its name suggests[1]), irrational numbers cannot exist precisely because they are "irrational", that is, inconsistent. And they are inconsistent because they have an actual infinite, and therefore inconsistent, number of decimal places. The only decimal numbers that can exist consistently must have a finite number of decimal places, and these are only the exact rational numbers. In other words, the only mathematically *real* numbers are the *exact rational* numbers. The algorithms that define irrational numbers, and periodic and mixed periodic rational numbers (division of two integers), actually define potentially infinite sequences of exact rational numbers. This will undoubtedly be a mathematical relief for rational intelligence.

2. Definitions and fundamental results

Remember that a binary relation between the elements of a set A can have, among others, the following properties:

1. Irreflexive: $\forall a \in A$: not $a < a$.
2. Asymmetric: $\forall a, b \in A$: If $a < b$ then not $b < a$.
3. Transitive: $\forall a, b, c \in A$: If $a < b$ and $b < c$, then $a < c$.
4. Dense: $\forall a, b \in A$: $\exists c$: $a < c < b$.

Now consider the following fundamental definitions and results that will be used in the next sections:

Definition 1 (of Strictly Ordered Sets) *A set is strictly ordered if a binary relation between its elements can be defined that is irreflexive, asymmetric, and transitive.*

[1] Lacking sound judgment or logic.

Definition 2 (of densely ordered sets) *A strictly ordered set is also densely ordered if the binary relation defining its strict order is also dense.*

Definition 3 (of Successors and Predecessors) *In strictly ordered sets, all elements that, in the ordering of the set, follow (precede) a given element of the set, are its successors (predecessors). If between the given element and one of its successors (predecessors) there is no other element, then this successor (predecessor) is the immediate successor (predecessor) of the given element.*

Definition 4 (of Complete Totality) *A complete totality is a set in which every element that satisfies the corresponding membership definition of the set is in the set.*

In consequence, to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains *all of them.*

Definition 5 (of Cardinal numbers) *The cardinal of a set is the exact number of elements of the set.*

Definition 6 (of the Types of Sets) *A set is finite if it has a definite and finite number of elements, i.e. a finite cardinal. A set of elements of a certain type is potentially infinite if it always contains a finite number of elements of that type and any finite numbers of new elements of that type can always be added to it, without the set ceasing to be finite and without it being necessary to change its name or symbol.*

Compare the above definition of potentially infinite sets with the following definition of actual infinite set based on R. Dedekind's classical definition [2, p. 115]:

Definition 7 (of the Actual Infinite Sets) *A set is infinite if it is a complete totality that can be put into one-to-one correspondence with one of its proper subsets.*

The demonstrations of the following three results can be seen, for example, in [4]

Corollary 1 (of the Inconsistent Infinite Sets) *All infinite sets are inconsistent.*

Corollary 2 (of the Inconsistent Axiom of Infinity) *The axiom of infinity is inconsistent.*

Theorem 1 (of the Actual Infinity) *The actual infinity is inconsistent.*

And from them, it is immediate to prove the following three corollaries:

Corollary 3 (of the Types of Sets) *All sets are either finite or potentially infinite.*

Proof.-This is an immediate consequence of Definition 6 and Corollary 1. □

Corollary 4 (of the Potentially Infinite Sets) *Potentially infinite set do not have a definite cardinal.*

Proof.-It is an immediate consequence of Definitions 6 and 5. □

Corollary 5 (of the Finite Number of Digits) *No number can have an actual infinite number of digits, either in the integer part or in the decimal part.*

Proof.-If a number had an actual infinite number of digits in its integer part or in its decimal part, then the set of those digits would be an actual infinite set, and therefore an inconsistent set according to Corollary 1. □

The rest of the article also assumes the four basic arithmetic operations and their fundamental properties in all numerical sets.

3. The set N of the natural numbers

The natural numbers are usually defined in more or less informal terms as the numbers used to count the quantity of objects in any group of objects: 1, 2, 3, ... And since the same group of objects cannot have two different quantities of objects at the same time, the natural number that counts them can only have a single value determined by that group of objects. They are also defined as those that satisfy the five classical Peano axioms, or other modern versions of them. The fifth of these axioms establishes the existence in the act of all natural numbers in the same set: the set of all natural numbers, which is therefore an actual infinite set. And therefore an inconsistent set according to Corollary 1. Here, only three axioms will be used, and some very basic results related to fundamental finitism will be demonstrated.

On the other hand, expofactorial and n-expofactorial numbers are remembered to illustrate how incredibly large natural numbers can become, so large that they are actually useless for describing the physical world. And as will be seen in the next sections, natural numbers define the rest of the numbers: integers numbers (including 0) and rational numbers. As will be proved, the set of real number coincides with the set of rational numbers, so that the set of real numbers is unnecessary. This is undoubtedly the most significant consequence of set-theoretic finitism. In this finitist frame, the axioms that legitimize and characterize natural numbers could be the following three:

Axiom 1 (of the First Natural Number) *The number 1 is the first natural number; It is the smallest of all natural numbers.*

Axiom 2 (of the Natural Precedence) *Each natural number n has a unique value, a unique immediate successor $n + 1$, and, except for 1, a unique immediate predecessor $n - 1$.*

Axiom 3 (of the Three Natural Numbers) *If m, n, and p are three natural numbers that satisfy: $m + n = m + p$, then $n = p$.*

According to the above axioms, it is possible to define the natural order of precedence of the natural numbers 1, 2, 3, ... in which each natural number n has an immediate successor $n+1$ and, except for 1, an immediate predecessor $n - 1$. All of which allows us to prove the following results:

Theorem 2 (of the Natural Order) *The set $\mathbb{N}$ of natural numbers in their natural order of precedence is strictly ordered but not densely ordered.*

Proof.-The set $\mathbb{N}$ of natural numbers in its natural order of precedence verifies the properties:

1.- *Irreflexive*: No natural number n satisfies $n < n$, otherwise the natural number n would have two different values, one less than the other, which is impossible (Axiom 2 of the Natural Precedence).

2.- *Asymmetric*: No pair m and n of natural numbers satisfies $m < n$ and $n < m$. If this were not the case and $m < n$ and $n < m$ were satisfied, adding the two inequalities member by member and applying the commutativity of addition, we would obtain $m + n < m + n$, which is impossible due to the irreflexive property.

3.- *Transitive*: $m < n < p$ implies $m < p$. If this were not the case, it would be $m < n < p \leq m$ and therefore $m < m$, which is impossible due to the irreflexive property.

Therefore, the set $\mathbb{N}$ is strictly ordered (Definition 1). And taking into account that between any natural number n and its immediate successor $n+1$ (Axiom 2) there is no other natural number (Definition 3), the set $\mathbb{N}$ of the natural numbers in their natural order of precedence cannot be densely ordered (Definition 2). □.

Theorem 3 (of the Number of Predecessors) *Every natural number n > 1 has n-1 predecessors in the natural order of precedence of the natural numbers.*

Proof.-In the natural order of precedence of natural numbers, the number 1 has no predecessor because it is the first natural number and the smallest of them all (Axiom 1 of the First Natural Number). The natural number 2 has exactly $2 - 1$ predecessors (the number 1). Suppose that a natural number m has $m-1$ predecessors; the immediate successor of m, which is the number $m+1$ (Axiom 2), will have m predecessors: the $m-1$ predecessors of m, plus m itself. So that $m+1$ has $(m+1)-1$ predecessors. Therefore, it can be stated inductively that every natural number $n>1$ has exactly $n-1$ predecessors. □

Theorem 4 (of the Cardinal Numbers) *Every natural number n is the cardinal number of the set of the first n natural numbers in their natural order of precedence.*

Proof.-The set of the first n natural numbers in their natural order of precedence has a first element, the number 1, and a last element, the number n, which has exactly $n-1$ predecessors in that set (Theorem 3 of the Number of Predecessors). Therefore, the set of the first n natural numbers in their natural order of precedence has exactly n elements: the $n-1$ predecessors of n plus n itself. Therefore, n is the cardinal of the set of the first n natural numbers in their natural order of precedence (Definition 5). □

Theorem 5 (of the Finite of Natural Numbers) *All natural numbers are finite, and do have a finite number of digits.*

Proof.-If a natural number n were infinite, the cardinal of the set of the first n natural numbers would be infinite (Theorem 4) and therefore inconsistent (Theorem 1). Therefore, no natural number can be infinite. And being a type of number, they can only have a finite number of digits (Corollary 5). □

Theorem 6 (of the Potential Infinitude of N) *The set $\mathbb{N}$ of the natural numbers is potentially infinite.*

Proof.-If $\mathbb{N}$ were actual infinite, it would be inconsistent (Theorem 1). If it were finite, it would have a certain finite number n of elements, as is the case with the set N_n of the first n natural numbers in their natural order of precedence, whose cardinal is n (Theorem 4 of the Cardinal Numbers). But n has an immediate successor $n+1$ (Axiom 2) that is not in the set N_n, which only contains the first n natural numbers. Therefore, $\mathbb{N}$ cannot contain only n elements. And the same argument holds for any natural number n. Therefore, the set $\mathbb{N}$ must be potentially infinite and have not a definite cardinal (Definition 6). □

Theorem 7 (of the Number of Successors) *Every natural number has a potentially infinite number of successors in the natural order of precedence of natural numbers.*

Proof.-Let a and n be any two natural numbers. If a had a finite number n of successors in the set $\mathbb{N}$ of the natural numbers in their natural order of precedence, this set $\mathbb{N}$ would have a definite cardinal $a+n$, which is impossible (Corollary 4 of the Potentially Infinite Sets). Therefore, the number of successors of any natural number in the set of the natural numbers in their natural order of precedence can only be potentially infinite. □

Theorem 8 (of the Incomplete Totality of $\mathbb{N}$) *The set $\mathbb{N}$ of the natural numbers is not a complete totality.*

Proof.-It is an immediate consequence of Definitions 4 and 6, and the Theorem 6 of the Potential infinitude of the set $\mathbb{N}$.

On another note, I will conclude this section by pointing out that it is not very common to find works in primary and secondary mathematics literature on the incredibly large values that natural numbers (all of which are finite) can reach. This is the case, for example, with expofactorial $n^!$ and n-expofactorial $n^{!m}$ numbers defined by the author in [3]. These are finite numbers whose written expression in normal text (say, 5 mm per digit) is greater than the diameter of the observable universe (about 90 billion light-years).

A finite set of natural numbers in their natural order of precedence can obviously have any finite natural number as its cardinal, for example the number 100-expofactorial of 100 (symbolically written $100^{!100}$), whose expression written in normal text would surely take up millions of times more space than the diameter of the observable universe (I have not been able to calculate it due to the limitations of my calculator and the calculators available online). These sets and numbers, on the other hand, seem to me to be completely useless for explaining the physical world, although they illustrate the inconsistent foolishness of infinite cardinal numbers.

4. The monary numbering system

The monary system of representing natural numbers is the simplest and oldest of these representation systems: it uses only the digit 1. In this system, the successive natural numbers are represented as:

$$1,\ 11,\ 111,\ 1111,\ 11111,\ 111111, ... \tag{1}$$

Naturally, this is very inconvenient for writing and, above all, for calculation. But it expresses very well the idea that each natural number is exactly one unit greater than its immediate predecessor. And also, as we will see, to demonstrate the fallacy that there are *infinitely* many *finite* natural numbers, each one unit greater than its immediate predecessor, and all of them mathematically existing in the act (actual infinity).

The following supertask calls into question this assumption of the actual infinity. Indeed, let $\{t_i\} = t_1, t_2, t_3 \ldots$ be a strictly increasing and convergent sequence of time instants within the finite open real interval of time (t_a, t_b), where t_b is the limit of that sequence. And consider the following supertask S: at each successive instant t_i of $\{t_i\}$, the single monary digit 1 is added to an initially empty string M. And it is added if, and only if, the resulting string of 1s is the expression of a finite natural number in the monary numbering system. Therefore, at the successive instants of $\{t_i\}$ we will have the successive natural numbers $1,\ 11,\ 111, 1111,\ 11111,\ \ldots$. Since t_b is the limit of $\{t_i\}$, that instant t_b is the first instant after all the instants of $\{t_i\}$. So at t_b the supertasks S will be finished.

According to the condition imposed on the supertask S, the resulting string M, although unknown, can only be the monary expression of a finite natural number. Let n be any finite natural number. It is clear that M cannot be the monary expression of the natural number n, because $n+1$ is also a natural number and the $n+1$-th digit 1 was added to M, just at the precise instant t_{n+1} of $\{t_i\}$. And since n is any natural number, it must be concluded that M cannot be the monary expression of any natural number n. Consequently, it must be concluded that, at the instant t_b, the string M is, and is not, the monary expression of a natural number. This is a contradiction derived from the hypothesis of the actual infinity, which assumes the existence in the act of each and every one of the infinitely many instants of the sequence $\{t_i\}$, and, of course, the existence in the act of the complete and actually infinite sequence of all finite natural numbers (the set $\mathbb{N}$ of all natural numbers, whatever the numbering system used).

5. The set *Z* of the integer numbers

As will be seen in this section and the next, natural numbers define all other numbers, which, for the reasons that will be given, are only integers and rational numbers. With regard to the set of integers, its elements can be defined in arithmetical terms as follows:

Definition 8 *The elements z of the set $\mathbb{Z}$ of integer numbers are the numbers resulting from subtracting any two natural numbers.*

Consequently, we would have: $\forall z \in \mathbb{Z}:\ z = m - n;\ m, n \in \mathbb{N}$. Definition from which result:

Positive integers if $m > n$,

The number zero if $n = n,\ \forall n \in \mathbb{N}$,

Negative integers if $m < n$.

Therefore, the set Z of integers in their natural order of precedence will be:

$$Z = \{... - 3, -2, -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\} \tag{2}$$

where the dots '...' at both ends mean that the set $\mathbb{Z}$ is potentially infinite in both directions, that of negative integers (preceded by the sign -) and that of positive integers.

For the same reasons as in the case of the set of natural numbers (Theorem 2 of the Natural Order), the set of integers in their natural order of precedence, expressed in 2, is strictly ordered but not densely ordered. And also for the same reasons as in the case of the set of natural numbers (Theorem 5 of the Finite Natural Order), the integer numbers do have a finite number of digits, and their value can only be finite.

Similarly, readers can easily demonstrate that every integer number z in the set $\mathbb{Z}$ of integer numbers in their natural order of precedence, has a potentially infinite number of successors and a potentially infinite number of predecessors. Therefore, the set Z of integers is potentially infinite, in this case without a first element and without a last element. And for the same reasons as in the case of the natural numbers, the set $\mathbb{Z}$ of integer numbers in their natural order of precedence is not a complete totality.

6. The set $\mathbb{Q}$ of the rational numbers

With rational numbers, an important new feature appears: they all have an integer part, and many of them also have a decimal part, which represents any fraction of the numerical unit (i.e., of the natural number 1). The inconsistency of the actual infinity imposes limits on the number of digits in the decimal part, for the same reason that it also imposes limits on the number of digits in its integer part (Theorem 5 of the Finite Natural Order), and as will be seen below, this formal conclusion leads to very important differences in relation to the infinitist (actual infinity) version of the set of rational numbers and the set real numbers. Let us begin, then, by formally defining rational numbers from a finitist perspective:

Definition 9 (of Rational Numbers) *Every rational number q is the result of dividing an integer z, positive or negative, by the number 10 raised to an exponent n that can be zero or any positive integer:*

$$\forall q \in \mathbb{Q} : q = \frac{z}{10^n};\ z \in \mathbb{Z},\ n \in \{0, 1, 2, 3, \dots\} \tag{3}$$

It is immediate to demonstrate that the set $\mathbb{Q}$ is densely ordered. Indeed, in their natural order of precedence, the elements of this set verify the following properties:

Irreflexive: let $q_1 = z_1/10^n$ be any rational number. Suppose that:

$$q_1 = \frac{z_1}{10^n} < \frac{z_1}{10^n} \tag{4}$$

Multiplying both sides of the inequality by 10^n we would have:

$$z_1 < z_1 \tag{5}$$

which is impossible because z_1 is an integer number and the set Z of integer numbers satisfies the irreflexive property.

Asymmetric: let $z_1/10^m$ and $z_2/10^n$ be any two rational numbers, the first being less than the second. Suppose that the first were also greater than the second, we would have:

$$\left(\frac{z_1}{10^m} < \frac{z_2}{10^n}\right) \wedge \left(\frac{z_1}{10^m} > \frac{z_2}{10^n}\right) \tag{6}$$

Multiplying both sides of the two inequalities by 10^{m+n} we would obtain:

$$(10^n z_1 < 10^m z_2) \wedge (10^n z_1 > 10^m z_2) \tag{7}$$

which is impossible because $10^n z_1$ and $10^m z_2$ are two integers that therefore satisfy the asymmetric property.

Transitive: let $z_1/10^m$, $z_2/10^n$ and $z3/10^p$ be any three rational numbers, the first being less than the second, and the second less than the third:

$$\frac{z_1}{10^m} < \frac{z_2}{10^n} < \frac{z_3}{10^p} \tag{8}$$

Multiplying the three sides of the two inequalities by 10^{m+n+p} we obtain:

$$10^{n+p} z_1 < 10^{m+p} z_2 < 10^{m+n} z_3 \tag{9}$$

which are three integer numbers that, because they are integers, satisfy the transitive property. Therefore, we can write:

$$10^{n+p} z_1 < 10^{m+n} z_3 \tag{10}$$

and dividing both sides of the inequality by 10^{m+n+p}, we get:

$$\frac{z_1}{10^m} < \frac{z_3}{10^p} \tag{11}$$

And consequently, the transitive property is verified for any trio of rational numbers.

Dense: Let $q_1 = z_1/10^m$ and $q_2 = z_2/10^n$ be any two rational numbers such that $q_2 - q_1 > 0$. From these, we define the rational number $q_3 = 1/2(q_1 + q_2)$:

$$q_3 = \frac{z_1}{2 \times 10^m} + \frac{z_2}{2 \times 10^n} \tag{12}$$

Elementary arithmetic allows us to write:

$$q_3 - q_1 = \frac{z_1}{2 \times 10^m} + \frac{z_2}{2 \times 10^n} - \frac{z_1}{10^m} = \frac{z_2}{2 \times 10^n} - \frac{z_1}{2 \times 10^m} = \frac{1}{2}(q_2 - q_1) > 0 \tag{13}$$

$$q_2 - q_3 = \frac{z_2}{10^n} - \frac{z_1}{2 \times 10^m} - \frac{z_2}{2 \times 10^n} = \frac{z_2}{2 \times 10^n} - \frac{z_1}{2 \times 10^m} = \frac{1}{2}(q_2 - q_1) > 0 \tag{14}$$

Therefore: $q_1 < q_3 < q_2$. Consequently, for every pair of rational numbers q_1, q_2 such that $q_1 < q_2$, there exists a rational number q_3 that satisfies $q_1 < q_3 < q_2$. As a result, we can state that the set $\mathbb{Q}$ of rational numbers is densely ordered.

Thus, in the set $\mathbb{Q}$ of rational numbers, between each two of its elements there always exists a potentially infinite number of other different rational numbers. Consequently, the symbolic expression of that set in its natural order of precedence can only be of the type:

$$\mathbb{Q} = \{\cdots - 3 \cdots - 2 \cdots - 1 \ldots 0 \ldots 1 \ldots 2 \ldots 3 \ldots\} \tag{15}$$

It is time to write down the only three possible numerical sets in finitist mathematics: the set of natural numbers $\mathbb{N}$, the set of integer numbers $\mathbb{Z}$, and the set of rational numbers $\mathbb{Q}$, all ordered in their natural order of precedence, which should serve to appreciate the essential role of natural numbers and the potential infinity, represented by "...", in all numerical sets:

$$\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\} \tag{16}$$

$$\mathbb{Z} = \{\ldots\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\} \tag{17}$$

$$\mathbb{Q} = \{\cdots - 3 \cdots - 2 \cdots - 1 \ldots 0 \ldots 1 \ldots 2 \ldots 3 \ldots\} \tag{18}$$

The set $\mathbb{R}$ of real numbers is unnecessary: from a finitist point of view, it is indistinguishable from the set $\mathbb{Q}$ of rational numbers. As noted above, the algorithms defining periodic rational and irrational numbers really define potentially infinite sequences of exact rational numbers. And remember also that an exact rational number can have so many decimal places that writing it out in normal text would take up much more space than the diameter of the observable universe. This is the case, for example, with the exact rational number:

$$\frac{12345}{10^{100^{!100}}}$$

where $100^{!100}$ is the 100-expofactorial of 100 (see above and [3, Chp. 19]).

7. Conclusions

The numerical consequences of the inconsistency of the actual infinity on numerical sets are truly ironic: irrational numbers (as the main meaning of their name suggests and warns us) cannot exist precisely because they are "irrational" in the sense of inconsistent. Indeed they would have to have an actual infinite number, and therefore an inconsistent (i.e. irrational) number of decimal places. Therefore, the set of real numbers, whose only difference from the set of rational numbers is that it contains the impossible irrational numbers, is an impossible mathematical set, i.e. a mathematical unreal set, not a mathematical real one. The only numbers with decimals are the exact rational numbers. And the only consistent and ordered numerical sets in their natural order of precedence are the potentially infinite sets of natural numbers $\mathbb{N}$, of integer numbers $\mathbb{Z}$, and of rational numbers $\mathbb{Q}$ as defined above, which verify among themselves:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \tag{19}$$

It is curious and ironic that the only numbers that cannot exist were named irrational numbers (because they are different from any "ratio" between two integers [1, p. 310]), and that irrational numbers also means inconsistent numbers, which is really what they finally are because of the assumed actual infinite number of its decimal places.

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