the general science journal

The General Science Journal. September 2025.

Numerical Consequences Of The Inconsistent Infinity

Antonio León.

Retired Professor. Independent researcher in the foundations of science.

Abstract.-This article analyzes the formal consequences of the inconsistent actual infinity on certain types of numbers and numerical sets. These results are so strong that I have not included them in this abstract to prevent some readers from abandoning their careful reading of the article at this point. They are the results that can be expected when the existence of a formal object that has always been discussed — the actual infinite — is axiomatically accepted without discussion for over a century. A type of infinity whose inconsistency can be demonstrated simply by using the tools provided by the new infinitist mathematics built upon it.

Keywords: actual Infinity, potential infinity, axiom of infinity, complete totality, natural numbers, integer numbers, rational numbers, irrational numbers, real numbers.

A revolutionary idea needs space to breathe, but academia often stifles that breath with the weight of conformism.
David Bohm, quoted in [2].

1. Introduction

Reversing contemporary infinitist mathematics will not be an easy task, which is understandable to a certain extent when one considers that it has been established in both contemporary mathematics and physics for over a century, with hardly any discussion. The history of science is full of similar cases. But in the case of infinity, it is surprising that discussions about its actual or potential nature, which had been going on for more than twenty centuries, ceased so suddenly and radically at the beginning of the 20th century. And not because the inconsistency of potential infinity had been proven. The Axiom of Infinity (whose infinity can only be the actual infinity [4]) was simply proposed, accepted, and since then (the early 20th century) all discussions about the actual or potential nature of infinity have abruptly ended in favour of the actual infinity. Although there were authors of the intellectual stature of H. Poincaré, L. Wittgenstein, or H. Weyl, among some others, who expressed their disagreement [5].

Discussing the Axiom of Infinity was not part of my initial scientific goals. Much less dedicating more than forty years to that discussion. But that is what actually ended up happening. Interested readers can find here [3] a summary of my research on the inconsistent nature of the actual infinity; and here [4, Pdf] a brief demonstration of that same inconsistency. I think it is worth spending ten minutes reading the article I have just referred the reader to. If you are not convinced of the inconsistency of the actual infinity, you can save yourself the trouble of reading the rest of this article. Otherwise, I would like to point out that the content of this article is devoted to the formal consequences that the inconsistency of the actual infinity has on numbers and on all basic numerical sets. In fact, this inconsistency affects all numerical sets considered as complete infinite totalities, as well as to the very existence of irrational numbers and rational numbers with an infinite real number of digits.

2. Preliminary considerations

I just used the expression "*complete totality*," which is not very common in mathematical literature, so I will recall here the definition given in [4]:

Definition 1 (of Complete Totality) *A complete totality is a set defined by comprehension in which every element that satisfies the corresponding membership definition of the set is in the set.*

In consequence, to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains *all of them*. It is now possible to give the definition of finite set and potentially infinite set, the latter being completely ignored by contemporary infinitist mathematics. [4]:

Definition 2 (of the Types of Sets) *A set is finite if it has a definite and finite number of elements. A set of elements of a certain type is potentially infinite if it always contains a finite number of elements of that type and any finite numbers of new elements of that type can always be added to it, without the set ceasing to be the same potential infinite set of the same type of elements, and therefore without having to change its name.*

Consequently, a potentially infinite set cannot be a complete totality. On the contrary, the actual Infinite set whose existence establishes the Axiom of Infinity, is assumed to be a complete totality. Finally, the following theorem, taken from [3, p. 54], establishes the finiteness conditions for finite sets:

Theorem 1 (of the Finite Sets) *If a set has a first element, a last element, and each element, except the last, has an immediate successor and, except the first, an immediate predecessor, the set has a finite number of elements.*

Proof: Let $X = \{a, b, c, \ldots v\}$ be a set with a first element a, a last element v and such that every element, except v has an immediate successor and, except a, an immediate predecessor. The immediate successor of a has a finite number of predecessors: 1 predecessor, just the element a. Suppose that, being h any element of X different from a and v, that element h has a finite number n of predecessors. The immediate successor of h has one more predecessor than h, the element h itself. Therefore, it also has a finite number $n+1$ of predecessors. (Peano's Axiom of the Successor [6, p. 1]). Since the immediate successor of a has a finite number of predecessors, we can inductively conclude that, except a and b, every element of X has a finite number of predecessors. And since a has no predecessors and v has one predecessor more than its immediate predecessor, the number of predecessors of v is also finite (Peano's Axiom of the Successor [6, p. 1]). Therefore, the number of elements of X, which is 1 plus the number of predecessors of its last element v, is finite (Peano's Axiom of the Successor [6, p. 1]). $\square$

On the contrary, and as already formally indicated (Definition 2), a potentially infinite set is a set with a finite number of elements to which it is always possible to add a finite number of new elements that are not in the set, without the set ceasing to be the same potentially infinite set of the same type of elements, and therefore without having to change its name.

3. Natural numbers and integer numbers

It is worth recalling here the axiomatic definition of natural numbers (Peano's axioms) because these numbers are directly part of the definition of all the other types of numbers:

1.- Zero is a natural number that is not the successor of any other natural number.

2.- Every natural number has a unique successor.

3.- Two natural numbers with the same successor are the same number.

4.- Every property that holds for zero and is inherited from a number to its successor holds for all natural numbers.

It should also be clarified that zero is not universally accepted as a natural number, but as an integer number. Since zero is not the successor of any natural number, there are no natural numbers smaller than zero. All of them are therefore equal to or greater than zero, so all of them are positive. Integer numbers do include negative numbers.

Unlike the other types of numbers discussed in the following sections, the existence of natural numbers and integer numbers, as such natural numbers and integer numbers, is not affected by the inconsistency of the actual infinity, since they are all finite numbers. What is obviously nullified is the existence of all actual infinite sets of these types of numbers. Therefore, only finite and potentially infinite sets of natural or integer numbers can exist.

The big difference between the finite sets and the potentially infinite sets is that in the case of the finite sets, it is not possible to add new elements that are not in the set without the set ceasing to be the same set it was before the addition. When adding new elements to a finite set you will always obtain a different set that will have to be renamed with a different name. But, as already indicated, this does not happen with the potentially infinite sets: in these sets, it is always possible to add new elements of the same type that are not in the set without the set ceasing to be the same set it was before the addition and, therefore, without having to change its name after the addition of the new numbers.

According to Theorem 1 of the Finite Sets, it is convenient to represent any finite set of numbers by writing its first element, its last element, and the rest of the elements between them; or just a small number of these elements between the first and the last of such elements, using ellipsis where appropriate and in such a way that it is clear which elements each ellipsis replaces, and always bearing in mind that each element must have an immediate predecessor (except the first) and an immediate successor (except the last).

As for the representation of potentially infinite sets of natural or integer numbers, it is not possible to include all their elements because these sets cannot exist as complete totalities (Definitions 1 and 2). Therefore, their representation must include one or more ellipses in places where it is always possible to add a finite number of new numbers of the same type that are not in the set without the set ceasing to be the same potentially infinite set after the addition of the new numbers.

In reality, finite sets, and therefore potentially infinite sets, can have an unimaginably large number of elements. Consider that there are natural, and then finite, numbers so large that writing them in normal text (e.g., 3mm per digit) would take up a distance millions of times greater than the diameter of the visible universe (ninety billion light-years). This is the case, for example, with expofactorial and n-expofactorial numbers [3].

4. Rational numbers

The definition of rational numbers (the elements of the set $\mathbb{Q}$) is given in terms of an elementary arithmetic operation (division) between any two elements of the set $\mathbb{Z}$ of integer numbers:

$$\forall q \in \mathbb{Q}: \ q = \frac{m}{n}; \ m, \ n \in \mathbb{Z} \tag{1}$$

As is well known, the result of these divisions gives rise to four types of rational numbers:

1.- Rational numbers without decimals (integers).

2.- Rational numbers with a finite sequence of successive decimal digits (exact rational numbers).

3.- Rational numbers with a finite sequence of successive decimal digits (period) that repeat successively and infinitely after the decimal point (pure periodic rational numbers)

4.- Rational numbers with a finite sequence of non-periodic decimal digits after the decimal point, immediately followed by a finite sequence of decimal digits (period) that repeat successively and infinitely (mixed rational numbers).

Unlike natural numbers and integer numbers, in the case of rational numbers, the inconsistency of the actual infinity has very surprising and significant consequences: a potential infinite number of rational numbers does not exist: pure periodic rationals and mixed periodic rationals. In fact, the division of integers that would supposedly give rise to these numbers (for example, $1/3$) gives rise to a potentially infinite sequence of different rational numbers, none of which is the number in question. For example:

$$\frac{1}{3} \approx 0.3 \approx 0.33 \approx 0.333 \approx 0.3333 \approx 0.33333 \approx 0.333333 \approx 0.3333333 \approx 0.33333333\ldots \tag{2}$$

Each of these rational numbers is a different exact rational number, for example:

$$0.33333 = \frac{33333}{100000} \tag{3}$$

but none of them is the exact result of the division between 1 and 3. Therefore, none of them is the supposed rational number 1/3. Furthermore, the above sequence, and the one originating from the division of integers that gives rise to any other periodic rational number, can only be potentially infinite: otherwise it would be inconsistent. For the same reason, the potentially infinite set $\mathbb{Q}$ cannot contain actual infinite subsets without being inconsistent. Therefore the set $\mathbb{Q}$ cannot contain all the terms of (2). This very important numerical restriction is valid for any other supposed periodic rational number:

Periodic rational numbers do not exist, neither pure nor mixed. The cor-

> responding divisions of integers give rise to potentially infinite sequences of distinct exact rational numbers, none of which is the one supposedly defined by the corresponding division. Therefore, only exact rational numbers with a finite number of decimal digits exist, which may include finite subsets of successive decimals that repeat successively a finite number of times, so that the result is always a rational number with a finite number of decimal digits; that is, an exact rational number.

The inconsistency of the actual infinity implies that the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ can only be potentially infinite: it is always possible to add new elements to these sets that were not previously in them. But these sets will continue to contain a finite number of numbers after each addition of new elements.

I would like to remind you once again of the unimaginable magnitude that finite natural numbers can reach: writing them in normal text would not fit within the diameter of the observable universe. And the same applies to the finite decimal strings of exact rational numbers: the written expression of most of them would not fit within the diameter of the observable universe. So, although from a theoretical and formal point of view the above conclusion is extremely important, from the point of view of the practical use of rational numbers, this finitist restriction has no consequences. In fact, we never use rational numbers with more than a few dozen of successive decimals for practical purposes. Physics, for example, never uses rational numbers with, say, millions of successive decimals.

5. Irrational numbers

Irrational numbers are also numbers that supposedly have infinite decimal digits after the decimal point, although in this case they are sequences of digits that are never periodic or mixed periodic. Therefore, none of them can be defined by simply dividing two integers. They can only be defined using more complex algorithms. Among irrational numbers, there are some very significant numbers that are widely used in both mathematics and physics, for example: π, e, Φ, $\sqrt{2}$, etc.

In accordance with the inconsistency of the actual infinity, the successive consistent approximations of the corresponding algorithms for calculating irrational numbers are always numbers with a finite number of decimal digits. For example, using Euler's algorithmic definition for the irrational number e:

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \dots \tag{4}$$

a potentially infinite sequence of numbers is obtained which, for the reasons given in the previous section, can only be exact rational numbers, none of which is the number e, although they approach its algorithmic definition (4) indefinitely:

$$e \approx 2.5 \approx 2.6666666 \approx 2.7083333 \approx 2.7166666 \approx 2.7180555 \approx 2.7182539 \approx 2.7182787 \dots \tag{5}$$

For the reasons given in the previous section, all these numbers must be exact rational numbers, i.e., rational numbers with a finite number of decimal digits. And the same applies to any other irrational number defined by any other algorithm. The numerical conclusion is extremely important:

> Irrational numbers (which seem to live up to their name) do not exist. Only

potentially infinite sequences of exact rational numbers exist, whose elements indefinitely approximate the corresponding algorithmic definitions of the supposed irrational numbers.

6. Real numbers

The set $\mathbb{R}$ of real numbers includes rational numbers and irrational numbers, and only them. But it has just been proven in the previous section (always in accordance with the inconsistency of the actual infinity) that irrational numbers do not exist. The algorithmic definition of any irrational number can only include a potentially infinite number of successive steps, none of which defines the supposed irrational number defined by that algorithm. The number of successive steps can always be increased, but it will always remain a finite number of steps; otherwise, it would be an inconsistent number of steps. And the result of each of these successive steps will always be an exact rational number with a finite number of decimal digits, because rational numbers with an infinite (actual) number of decimal digits cannot exist either.

Now, if real numbers exclusively include rational numbers and irrational numbers, and irrational numbers do not exist, then the set of the real numbers is identical to the set of the rational numbers. It must be concluded that, according to the inconsistency of the actual infinity, the only numerical sets that include different kinds of numbers are the set of natural numbers, the set of integer numbers, and the set of rational numbers. All three with the same type of infinity: the potential infinity. This is a devastating consequence for infinitist mathematics that assumes the Axiom of Infinity, an axiom that can no longer be assumed because it is inconsistent, as demonstrated in the 4 pages of [4].

On the other hand, the consequences for physics and for any other science or technique that makes use of numbers, are practically non-existent: numbers with, say, a thousand decimal digits are rarely used, an insignificant number compared to that of some integers and some exact rational numbers, whose writing in normal text, I repeat once again, does not fit within the diameter of the observable universe. The importance is purely mathematical, logical and philosophical. And also one of scientific hygiene and human humility.

Bibliographical References

[1] Richard Dedekind. *Qué son y para qué sirven los números (Was sind Und was sollen die Zahlen (1888))*. Alianza, Madrid, 1998. Definición de conjunto infinito p. 115.

[2] B. Del Medico. *David Bohm. Un hombre fuera de lugar*. Edizioni PensareDiverso, 2025.

[3] A. León Sánchez. *Infinity put to the test*. Amazon's Kindle Direct Publishing, 2023 (2021). PDF.

[4] A. León Sánchez. The Axiom of Infinity Is Inconsistent. *The General Science Journal*, 2024. PDF.

[5] Andreas W. Moore. *The Infinite*. Routledge, New York, 2001.

[6] Giuseppe Peano. *Arithmetices Principia. Nova Methodo Exposita*. Libreria Bocca, Roma, 1889.