

## SOLUTION OF THE PROBLEM OF MULTISECTING AN ANGLE

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### Introduction

The problem of multisectioning an angle (i.e. the problem of division of a given arbitrary angle into the given set of equal parts using only a compasses and an unmarked straightedge) is a modern geometrical problem. It arises as generalization of the classical problem in ancient Greek mathematics – the problem of trisectioning an angle (i.e. the problem of division of a given arbitrary angle into three equal parts using only a compasses and an unmarked straightedge). As is known, the result of centuries efforts on the solution of the famous problem of trisectioning an angle has been summed up in 19th century. In 1837 Pierre Wantzel has reduced the problem of trisectioning an angle to the solution of the equation  $x^3 - 3x - 2 \cos \alpha = 0$  and has proved that the problem of trisectioning an angle has no solution except for some special cases. However, the search of general solution is continued till now. The modern analysis of the works on the problem of trisectioning an angle shows unsuccessful attempts undertaken by mathematicians to solve the problem completely, is explained by two circumstances. First, the idea of the general approach to the problem has not been properly guessed. Secondly, the statement of the problem of trisectioning an angle as a particular case of the problem of multisectioning an angle has not been proposed. The purpose of the present work is to propose the idea of the general approach to the problem and to propose the general solution of the problem of multisectioning (in particular, trisectioning) an angle using only a compasses and an unmarked straightedge.

### 1. The theoretical solution of the problem of multisectioning an angle

As is known, operation of multiplication of the given angle by an integral number  $n = 1, 2, 3, \dots$  – direct operation – is always carried out, and operation of division of a given arbitrary angle into the given number of equal parts – inverse operation – represents known difficulty. In this connection, following idea of the approach to a geometrical problem of division of an angle arises. Operation of multiplication of one quantity should be connected with operation of division of other quantity by such relation that operation of multiplication of one quantity would lead to operation of division of other quantity. This idea is embodied, for example, into the theory and design of the proportional compasses used for practical division of a straight line segment into some equal parts. Thus, the idea of the general approach to the geometrical problem of multisectioning an angle is that the general solution of this problem must be based on the theory of proportions and on the theory of similarity of geometrical objects.

Analysis of the theory of similarity of geometrical objects leads to the following general statement of the problem of multisectioning an angle: taking into consideration the theory of

similarity of concentric circles, one should find the relation between quantity of radius of a circle, length of an arch of a circle, and quantity of the central angle, which describes multisectioning the central angle with multiplication of quantity of radius by integral number  $n = 1, 2, 3, \dots$ .

The theoretical solution of the problem of multisectioning an angle is based on following known geometrical propositions.

- 1) **Definition.** The angle formed by two radiuses of the same circle is called central angle.
- 2) **Theorem.** If two central angles in the same circle are equal, then the arches corresponding to them are equal.
- 3) **Theorem.** If two arches in the same circle are equal, then the central angles corresponding to them are equal.
- 4) **Definition.** The straight line segment connecting two points of a circle is called chord.
- 5) **Theorem.** Equal arches are tied by equal chords.
- 6) **Theorem.** Equal chords tie equal arches.
- 7) **Theorem.** If diameter is perpendicular to a chord, then it divides this chord and arches tied by this chord into half.
- 8) **Theorem.** Diameter passed through the middle of the chord is perpendicular to the chord and divides the arches tied by this chord into half.
- 9) **Theorem.** Diameter passed through the middle of the arch divides the chord which ties this arch into half and is perpendicular to this chord.
- 10) **Theorem.** Diameter passed through arbitrary point of the arch divides this arch and the chord which ties this arch into proportional parts.
- 11) **Definition.** Triangles are called similar if their angles are equal in pairs and the like sides are proportional.
- 12) The theorems of similarity of triangles and the theorems of similarity of concentric circles.
- 13) Arches of the given circle are measured by the arch on which central angle is leaned.
- 14) **Definition.** The ratio of length of an arch of a circle, for which given angle is central, to length of radius of this arch is called radian measure of angle.
- 15) If radius  $R$  of circle and radian measure  $\alpha$  of arch are known, then one can calculate length  $l$  of this arch. Really, in accordance with definition of radian measure,

$$\alpha = \frac{l}{R}, \text{ or } l = \alpha R.$$

16) Ratio of lengths of arches of two concentric circles is ratio of lengths of their radiuses for the same central angle  $\alpha$  :

$$\frac{l_i}{l_j} = \frac{R_i}{R_j}, \text{ or } \frac{l_i}{R_i} = \frac{l_j}{R_j}, \quad i, j = 1, 2, 3, \dots$$

where  $i, j$  are order numbers of concentric circles. In other words, the ratio of length of arch of circle to its radius does not depend on radius for the same central angle  $\alpha$ . Quantity of this ratio is changed if the central angle is changed.

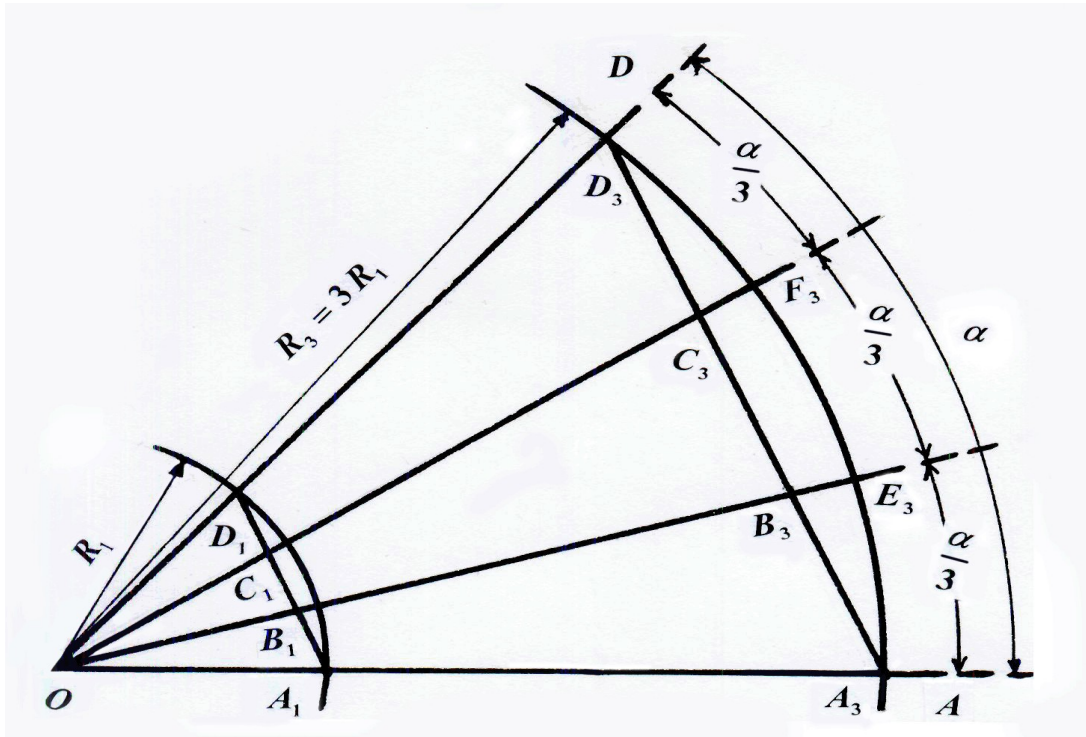
The proof of existence of the solution and the general solution of the problem of multisectioning an arbitrary angle  $\alpha = const$ , based on the theory of similarity of concentric circles, follow from these propositions. Really, if:

- (a) two circles designated by the order numbers 1 and  $n$  are given,
- (b) the operation of multisectioning of an angle  $\alpha$  is expressed by the ratios  $\alpha/n$ ,  $l_n/n = R_n \alpha/n$ , and  $R_1/R_n = 1/n$  where  $n = 1, 2, 3, \dots$ ,

then the relations  $l_1/l_n = 1/n$ ,  $\alpha/n$  take place for arbitrary  $R_1$ . In other words, if  $l_n/n = R_n \alpha/n$ ,  $R_n = nR_1$ , and  $l_n = nl_1$ , then length  $l_n$  of arch and angle  $\alpha$  are divided into  $n$  equal parts:  $l_n/l_1 = n$  and  $\alpha/n$  under arbitrary  $R_1$ . Since the multiplication operation  $R_n = nR_1$  is always carried out, the multiplication operation  $l_n = nl_1$  and, consequently, the division operation  $\alpha/n$  is always carried out as well. Thus, the obtained proof of the existence of a solution represents the general theoretical solution of the problem of multisectioning an arbitrary angle  $\alpha$ .

**2. The practical solution of the problem of multisectioning an angle**

The practical solution of the problem of multisectioning an arbitrary angle  $\alpha$  – the solution with only a compasses and an unmarked straightedge – is based on the above-stated theoretical proof. The example of the practical solution of the problem of trisecting an arbitrary angle  $\alpha$  is shown on the figure below.



The practical solution of the problem of trisecting an arbitrary angle  $\angle AOD = \alpha$

The practical solution of the problem of trisecting an arbitrary angle  $\alpha$  consists of following operational steps:

- 1) Draw an arbitrary angle  $\angle AOD = \alpha$ , where the point  $O$  is the vertex of the angle.
- 2) Draw the first (basic) circle of an arbitrary radius  $R_1$ , where  $n = 1$  is order number of the circle,  $O$  is the circle centre.
- 3) Draw the second (concentric) circle of the radius  $R_3 = 3R_1$ , centered in the point  $O$ , where  $n = 3$  is order number of the circle. Radii  $R_1$  and  $R_3$  are connected by the relation  $R_n/R_1 = n$ ,  $n = 3$ .
- 4) Designate the points of intersections of circles and of sides of angle  $\angle AOD$  by the letters  $A_1, D_1$  and  $A_3, D_3$ .
- 5) Draw the chords  $\overline{A_1D_1}$  and  $\overline{A_3D_3}$ . These chords are parallel each other.
- 6) Find the following proportions from similarity of triangles  $\Delta A_1OD_1$  and  $\Delta A_3OD_3$ :

$$\frac{\overline{A_3O}}{\overline{A_1O}} = \frac{\overline{OD_3}}{\overline{OD_1}} = \frac{\overline{A_3D_3}}{\overline{A_1D_1}} = \frac{R_3}{R_1}.$$

7) In consecutive order, set down three segments  $\overline{A_1D_1}$  on chord  $\overline{A_3D_3}$  using compasses. Designate the formed points by letters  $B_3$  and  $C_3$ .

8) Pass the segments  $\overline{OE_3}$  and  $\overline{OF_3}$  through the points  $B_3$  and  $C_3$ .

9) From the proportions

$$\frac{\cup A_3D_3}{\cup A_1D_1} = \frac{R_3}{R_1} \quad \text{and} \quad \frac{\overline{A_3D_3}}{\overline{A_1D_1}} = \frac{R_3}{R_1},$$

obtain the relation

$$\cup A_3D_3 = \left( \frac{\cup A_1D_1}{\overline{A_1D_1}} \right) \overline{A_3D_3}$$

where  $\left( \frac{\cup A_1D_1}{\overline{A_1D_1}} \right) > 1$  is proportionality coefficient. In accordance with formal logic, the qualitative determinacy of the left part of this relation is identical with the qualitative determinacy of the right part of this relation, i.e. the left and right parts of this relation belong to one and the same qualitative determinacy. This signifies that concepts “arch length  $\cup A_3D_3$ ” and “chord length  $\left( \frac{\cup A_1D_1}{\overline{A_1D_1}} \right) \overline{A_3D_3}$ ” are identical ones. Therefore, the relation

$$\frac{\cup A_3D_3}{3} = \left( \frac{\cup A_1D_1}{\overline{A_1D_1}} \right) \frac{\overline{A_3D_3}}{3}$$

defines correctly operation of division of the arch  $\cup A_3D_3$ : trisecting the arch  $\cup A_3D_3$  is trisecting the chord  $\overline{A_3D_3}$ . (In formal-logical sense, this mathematical relation defines identity of trisecting the arch  $\cup A_3D_3$  and of trisecting the chord  $\overline{A_3D_3}$ ).

10) Take into consideration that

$$\cup A_3E_3 = \cup E_3F_3 = \cup F_3D_3 = \frac{\cup A_3D_3}{3} \quad \text{under} \quad \frac{\overline{A_3D_3}}{3}.$$

This signify that

$$\angle A_3OE_3 = \angle E_3OF_3 = \angle F_3OD_3 = \frac{\angle AOD}{3} = \frac{\alpha}{3} \quad \text{under} \quad \frac{\overline{A_3D_3}}{3}.$$

Thus, the problem of trisecting an arbitrary angle  $\alpha$  has been solved. In the case of large angle  $\alpha$  (for example,  $\pi \leq \alpha \leq 2\pi$ ), one should divide the angle  $\alpha$  into two or four equal parts in order to use steps 1-10. The obtained solution of the problem of trisecting an angle  $\alpha$  signifies that multisectioning an arbitrary angle (i.e.  $\alpha/n$  under  $n > 3$ ) is a trivial procedure now.

### Conclusion

Thus, it is proved that the problem of multisectioning (in particular, trisecting) an arbitrary angle has a theoretical and practical solution in all cases. The proposed general solution of the problem is based on known geometrical propositions and, consequently, represents scientific truth.

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