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The matrix form of the hybrid-π model of the BJT

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Abstract: The small signal AC model for the hybrid-π equivalent circuit of the BJT is detailed. The circuit equations are cast into an elegant matrix form whose elements are accurately transformed into the h-matrix circuit model elements. To obtain the exact hybrid- π equations in matrix formulation one needs to consider in the network equations one more parameter, known as the feedback resistance r_μ between the base and collector terminals.The formalism is developed here first for the midband frequency range but can easily be extended also to the high frequency domain by exchanging the real conductances g_k with generalized reactances y_k.

Keywords: Hybrid-π model, BPT transistors, Equivalent Electronic circuits

1 Introduction

The 'simple' small-signal hybrid-π model, also called the Giacoletto Model [1], shown schematically in Fig. 1 is well known for decades, and appears in many texbooks [2,3]. It is still widely used also in analog electronic design for mid-band and high frequency calculations of circuits involving BPT transistors [3], HBT [4], BICMOS [5], OP-AMPS [6], FEEDBACK AMPS [7] and even multistage amplifiers [8,9]. In Fig. 1 B, C and E denote respectively the Base, Collector and Emitter of the BJT equivalent circuit.

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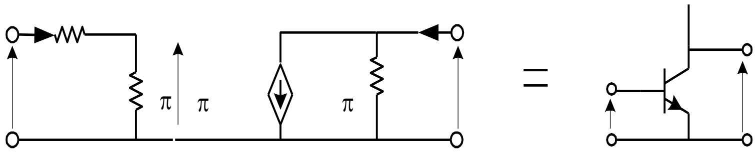

Fig. 1 – *The simple midband hybrid-p equivalent circuit of the BJT*

In this model, r_b is the BJT base resistance parameter, which can be calculated from the transistor base geometry and its electrical conductivity properties. The values of all other AC-dynamic resistive components can be estimated from the quiescent DC operating point Q(I_{CQ}, I_{BQ}) of the BJT, using proper dynamic models [3] giving in the simplest approximation the following relations:

$$r_O = \frac{V_{Early}}{I_{CQ}} = \frac{1}{g_o}; \qquad r_\pi = \frac{V_T}{I_{BQ}} = \frac{1}{g_\pi}; \qquad g_m = \frac{I_{CQ}}{V_T} = \frac{1}{r_m} \tag{1}$$

In Eqn(1), r_o and r_π denote respectively the dynamic output resistance and input resistance and g_m is termed as transconductance and sometimes confusingly, as mutual conductance [8,10]. I_{CQ} and I_{BQ} are the collector and base DC currents at the quiescent point and V_T and V_{Early} are the 'thermal' voltage and 'Early voltage' respectively [11]. The first parameter given in Eqn.(1) is just a simple approximation for the BJT output dynamic resistance, stemming from the base width modulation effect on I_C by V_{CE}, originating in the work of Early [12]. Hence by knowing the basic physical properties of the BJT and its operating DC or Q-point, it is possible in principle to calculate all the required hybrid-π parameters. One may recall also that $\beta = g_m r_\pi$ relates two parameters to β which itself is the well known dynamic current gain, a parameter of paramount physical importance in all BJT circuit models [11].

According to the over-simplified model, shown in Fig. 1, the BJT circuit equations for the midband small signal case are given by the following set of equations:

$$\begin{cases} v_{in} = i_{in}(r_b + r_\pi) \\ i_o = g_o v_o + g_m v_\pi \end{cases} \tag{2}$$

Unfortunately these simplified equations system used by electronic engineeers overwhelmingly in the linear dynamic incremental circuit analysis was never cast into a vectorial - matrix relation as one encounters in some well known two port electrical network models such as z, y, a, b, h and g [13,14].

In contrast, the h-matrix model, whose schematical circuit appears in Fig. 2, was cast long ago into a matrix – vectorial relation which connects the input voltage and output current column vector $\begin{pmatrix} v_i \\ i_o \end{pmatrix}$, with the input current and output voltage column vector $\begin{pmatrix} i_i \\ v_o \end{pmatrix}$, through the following well known matricial equation:

$$\begin{pmatrix} v_i \\ i_o \end{pmatrix} = \begin{pmatrix} h_i & h_r \\ h_f & h_o \end{pmatrix} \begin{pmatrix} i_i \\ v_o \end{pmatrix} \tag{3}$$

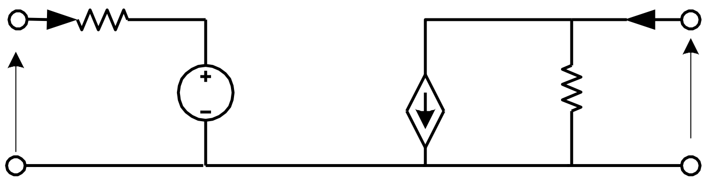

Fig. 2 – *The h-parameters model of the BJT*

The h-matrix elements have been preferred thus traditionally by the complete set of 4 hybrid parameters characterising the BJT. One can observe in particular that the h-hybrid model shown in Fig. 2 includes a parameter h_r which represents the feedback connection between the input and output stages. However, the simple, commonly used physical hybrid-π model described above does not include such a resistive feedback element in many texts [4,9], with the exception of [15] which discusses this shortcoming. Thus the transformation between these two models parameters is not accurate and gives only an oversimplified approximation given e.g., in the following set of equations:

$$\begin{aligned} h_i &\approx r_\pi + r_b \cong r_\pi \\ h_f &= r_\pi g_m \cong h_i g_m \\ h_o &= g_o \end{aligned} \tag{4}$$

One can thus conclude that the absence of a feedback parameter in the currently used too 'simplistic' midband hybrid-π model is the reason for the inability to obtain an accurate mathematical represention of the h_r and h_i parameters in terms of the hybrid-π model parameters. In what follows we show how the inclusion of the feedback dynamic resistance r_μ into the 'simplistic' hybrid-π model and carrying out a very simple but exact circuit analysis leads to an equations system which becomes both accurate and expressed matricially, as is the h-matrix model given for decades in all textbooks. It is possible therefore for the hybrid-π model to be written in an elegant matrix equation form which has also a deeper physical insight as one can see from the seminal works of Giacoletto [1] Middlebrook [16] and Sharma [17] which show for instance that a resistive element is essential between the base and collector to account for the charge carriers recombination in the control region of any active device. What we found also employing this feedback resistance in the new hybrid-π matrix formulation, is that each matrix element corresponds exactly to the hybrid-h matrix elements or parameters h_k given in eqn.(3) and Fig. 2. Also, the matrix form can be conveniently used for defining incremental models of BJT's, employing simple computerised circuit design programs, following the more elaborate though less intuitive 4-terminal matrix computerised models. In complex circuits which include energy storage elements, linear incremental models like the hybrid-π offer important advantages when compared to the physically lacking insight of SPICE or similar exact computer programs. This is particularly evident for developers which consider the dynamic behaviour of particular combinations of BJT's in new circuits and publish them in the literature. Thus the exact hybrid-π matrix formalism may enhance the understanding and increase the analysis accuracy in the still widely used circuit engineering practice which uses the hybrid-π model for the dynamic analysis of new complex circuits. These advantages are however obtained while still keeping an intuition stemming from the theory of the dynamics of linear passive circuits and physical insights of the BJT.

2 The feedback impedance

The impedance z_μ connecting the base and collector terminals may be generally used in the hybrid-π model at high frequencies to account for the capacitive Miller Effect [2,3,11]. In the 'simple' hybrid-π midband model this impedance degenerates into a simple dynamic resistance r_μ. The resistance value is commonly very high in comparison with other input and output parallel circuit resistances (but not allways) and is thus usually neglected. However, if one needs a better accuracy as happens in the case of high frequency, one needs to include this parameter in the calculations.

This BJT feedback dynamic resistance r_μ is sometimes defined by an incremental model as:

$$r_\mu \equiv \frac{\partial v_{CE}}{\partial i_B}\Big|_Q = r_o \cdot \beta \qquad (5)$$

where β stands for the small signal current gain of the transistor and r_o is given in eqn.(1), see details for instance in [11].

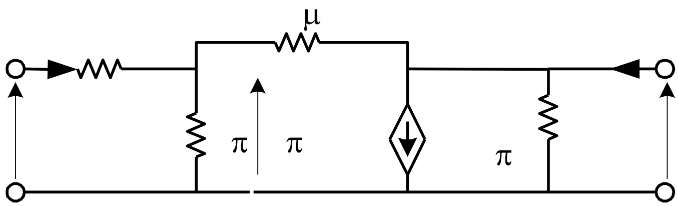

Fig. 3 – *The hybrid-p model scheme with the feedback resistance r_μ..*

To solve the improved full hybrid-π linear circuit given in Fig. 3 we may write the following circuit equations:

$$\begin{aligned}&(a)\ v_{in} = r_b i_{in} + v_\pi \\ &(b)\ i_{out} = v_{out} g_o + g_m v_\pi + (v_{out} - v_\pi) g_\mu \\ &(c)\ i_{in} = v_\pi g_\pi + (v_\pi - v_{out}) g_\mu\end{aligned} \qquad (6)$$

If we now eliminate v_π in the above set of equations using eqn. (6c) and inserting it in eqns. (6a) and (6c) we obtain the following set of equations:

$$\begin{cases} v_{in} = i_{in}\left(r_b + \dfrac{1}{g_\pi + g_\mu} \right) + v_{out}\left(\dfrac{g_\mu}{g_\pi + g_\mu} \right) \\ i_{out} = i_{in}\left(\dfrac{g_m - g_\mu}{g_\pi + g_\mu} \right) + v_{out}\left(g_o + \dfrac{g_\mu (g_\pi + g_m)}{g_\pi + g_\mu} \right) \end{cases}$$

One can cast now the above set of equations into a matrix form for the hybrid-π model of the BJT which to the best of our knowledge has never been published elesewhere before:

$$\begin{pmatrix} v_{in} \\ i_{out} \end{pmatrix} = \begin{pmatrix} r_b + \dfrac{1}{g_\pi + g_\mu} & \dfrac{g_\mu}{g_\pi + g_\mu} \\ \dfrac{g_m - g_\mu}{g_\pi + g_\mu} & g_o + \dfrac{g_\mu (g_\pi + g_m)}{g_\pi + g_\mu} \end{pmatrix} \begin{pmatrix} i_{in} \\ v_{out} \end{pmatrix} \qquad (7)$$

This matrix is very instructive, first from the physical point of view since it gives an intuitive feeling for the BJT various intrinsic conductances g_k ,where *k*

is the index for π, μ and o. Second, this new matrix is also identical to the h-matrix circuit, with elements that are measurable but consist of a non-physical model, though it operates exactly on the same vectors.
Third, this matrix model can also be easily expanded to the high frequencies domain by simply generalizing the g_k parameters into frequency dependable complex y_k parameters i.e., $y_k = g_k + b_k j$, giving us the following high frequency domain matrix of the hybrid-π model:

$$\begin{pmatrix} v_{in} \\ i_{out} \end{pmatrix} = \begin{pmatrix} r_b + \dfrac{1}{y_\pi + y_\mu} & \dfrac{y_\mu}{y_\pi + y_\mu} \\ \dfrac{y_m - y_\mu}{y_\pi + y_\mu} & y_o + \dfrac{y_\mu (y_\pi + y_m)}{y_\pi + y_\mu} \end{pmatrix} \begin{pmatrix} i_{in} \\ v_{out} \end{pmatrix} \tag{8}$$

In particular, using the transistor intrinsic capacitances - C_k to model the reactive device elements, such as diffusion and junctions depletion capacitances, one may substitute in the above matrix terms such as $y_k = g_k + j\omega C_k$ and use the hybrid-π matrix as an incremental convenient model for the high frequency domain of the BPT, i.e., up to about 500 MHz or even above. Physical inductances, L_k can also be used in the matrix elements given above for making it suitable for even higher frequencies such as the microwave frequency domain which is of current interest in cellular communication circuits.

3 Transforming accurately the hybrid–π matrix elements into h-matrix parameters

If we compare and equate the hybrid-π matrix elements to the h-hybrid matrix elements, the h parameters given usually by the manufacturers as measurable parameters [10], can be cast in terms of the hybrid-π physical parameters as:

$$\begin{matrix} h_i = r_b + \dfrac{1}{g_\pi + g_\mu} & h_r = \dfrac{g_\mu}{g_\pi + g_\mu} \\ h_f = \dfrac{g_m - g_\mu}{g_\pi + g_\mu} & h_o = g_o + \dfrac{g_\mu (g_\pi + g_m)}{g_\pi + g_\mu} \end{matrix} \tag{9}$$

By assuming $r_\mu \to \infty$ or $g_\mu \to 0$ usually taken in the midband frequency domain of the hybrid-π model we obtain the following approximations:

$$\begin{pmatrix} v_{in} \\ i_{out} \end{pmatrix} = \begin{pmatrix} r_b + \dfrac{1}{g_\pi + 0} & \dfrac{0}{g_\pi + 0} \\ \dfrac{g_m - 0}{g_\pi + 0} & g_o + \dfrac{0 \cdot (g_\pi + g_m)}{g_\pi + 0} \end{pmatrix} \begin{pmatrix} i_{in} \\ v_{out} \end{pmatrix}$$

$$\begin{pmatrix} v_{in} \\ i_{out} \end{pmatrix} = \begin{pmatrix} r_b + r_\pi & 0 \\ g_m r_\pi & g_o \end{pmatrix} \begin{pmatrix} i_{in} \\ v_{out} \end{pmatrix} \quad \Rightarrow \quad \begin{cases} v_{in} = (r_b + r_\pi) i_{in} \\ i_{out} = g_m v_\pi + g_o v_{out} \end{cases}$$

$$here \qquad v_\pi = r_\pi i_{in}$$

$$finally \quad giving: \begin{matrix} h_i = r_b + r_\pi & h_r = 0 \\ h_f = g_m r_\pi & h_o = g_o \end{matrix}$$

Hence we obtain in these approximations the particular case of ‘the simple’ hybrid-π equations aforementioned and their inexact transform equations used formerly to connect their parameters to the h-matrix model elements. Here however the last approximation is obtained from an exact circuit equations system. Thus, besides the mathematical rigor and elegancy of the improved matrix formulation, this correction improves practically also the derived h-matrix parameters calculated for BJT transistors. Hence alternatively to using the manufactures measured h_k parameters data, given usually for only a limited range of DC operable values and frequency dependent parameters we give in this paper a more exact and physically based model to analyze BJT’s. Finally, this matrix hybrid-π model formulation in eqn. (7) and eqn. (8) can be easily extended to other electronic devices such as FET, OPAMPS and FEEDBACK amplifiers [6,7,8,11].

4 Conclusion

By including the input - output feeback impedance into the hybrid-π circuit analysis of the BJT we obtained a system of equations leading to an elegant and enhanced physical circuit model matrix form, though only limited to the dynamic incremental case. As a result one can use a more accurate mathematical formalism to calculate the h-matrix parameters, from the physical π-hybrid parameters or vice-versa. Extending this method to the high frequency domain is also straightforward and should include possible poles and zeroes according to the degreee of complexity any specific BJT circuit may posess.

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